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Vol. XXXIII No. 2 February 2015

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Plot 99, Sector 44 Institutional Area, Gurgaon, (HR).
Tel : 0124-4951200
e-mail : info@mtg.in website : www.mtg.in

Regd. Office
406, Taj Apartment, Near Safdarjung Hospital,
Ring Road, New Delhi - 110 029.

Managing Editor : Mahabir Singh
Editor : Anil Ahlawat (BE, MBA)

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editorial

Vedic Mathematics and its applications in India

We are happy that at least now the contribution of our ancient mathematicians are being recognized. Ancient mathematics finds many applications in India. Architecture in building huge temple towers to the houses for the common man are all planned by our carpenters and masons.

Mathematical formulas are given in very short "Sutras" which the chief carpenter has to learn before he starts practising these in buildings. These rules, like the sanskrit grammar, are given in the form of short "Sutras" or formulae. Fast calculations are performed to determine the position of the planets with the help of sea-shells, even today.

Without naming "Group theory", it is common practice in India to display complicated "Rangolis" for various auspicious occasions particularly in the whole of South India from Maharashtra to Kanyakumari. The easiest way of learning group theory to interpret diffraction patterns is to practice the interrelations from 'Rangolis'. Our ancient learning is our precious heritage. We have to cherish our heritage and use them to push our frontiers of knowledge and to widen our vision.

Anil Ahlawat
Editor

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MATHS MUSING

Maths Musing was started in January 2003 issue of Mathematics Today with the suggestion of Shri Mahabir Singh. The aim of Maths Musing is to augment the chances of bright students seeking admission into IITs with additional study material. During the last 10 years there have been several changes in JEE pattern. To suit these changes Maths Musing also adopted the new pattern by changing the style of problems. Some of the Maths Musing problems have been adapted in JEE benefitting thousand of our readers. It is heartening that we receive solutions of Maths Musing problems from all over India.
Maths Musing has been receiving tremendous response from candidates preparing for JEE and teachers coaching them. We do hope that students will continue to use Maths Musing to boost up their ranks in JEE Main and Advanced.

Prof. Dr. Ramanaiah Gundala, Former Dean of Science and Humanities, Anna University, Chennai

PROBLEM Set 146

JEE MAIN

1. If $a = \cos\alpha + i\sin\alpha$ and the equation $az^2 + z + 1 = 0$ has a pure imaginary root, then $\tan\alpha =$

(a) $\frac{\sqrt{5}-1}{2}$ (b) $\frac{\sqrt{5}+1}{2}$ (c) $\sqrt{\frac{\sqrt{5}-1}{2}}$ (d) $\sqrt{\frac{\sqrt{5}+1}{2}}$

2. $\int_1^e x^2(\ln x)^3\,dx =$

(a) $\frac{2}{27}(2e^3+1)$ (b) $\frac{2}{9}(2e^3+1)$

(c) $\frac{2}{9}(2e^3-1)$ (d) $\frac{2}{27}(2e^3-1)$

3. If N is the coefficient of x^2 in the expansion of $(1-x)(1+3x)(1-5x)(1+7x)\ldots(1-17x)(1+19x)$, then the sum of the digits of $|N|$ is

(a) 10 (b) 11 (c) 12 (d) 13

4. Let S be the sum of all recurring decimals $x = 0.\overline{abc} = 0.abcabc\ldots$, where a, b, c are distinct digits. The sum of the digits of S is

(a) 5 (b) 7 (c) 9 (d) 10

5. In a triangle ABC, if $C = \frac{\pi}{6}$ and $a = x^2 + x + 1$, $b = x^2 - 1$, $c = 2x + 1$, then $x =$

(a) $2+\sqrt{3}$ (b) $1+\sqrt{3}$

(c) $4\sqrt{3}$ (d) $-(2+\sqrt{3})$

JEE ADVANCED

6. If $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$, $\vec{c} = \hat{i} + \hat{j} - 2\hat{k}$, then a vector in the plane of $\vec{b}$ and $\vec{c}$ whose projection on $\vec{a}$ is of magnitude $\sqrt{\frac{2}{3}}$ is

(a) $2\hat{i} + 3\hat{j} - 3\hat{k}$ (b) $2\hat{i} + 3\hat{j} + 3\hat{k}$

(c) $-2\hat{i} - \hat{j} + 5\hat{k}$ (d) $2\hat{i} + \hat{j} + 5\hat{k}$

COMPREHENSION

Let P be a variable point on the ellipse $x^2 + a^2y^2 = a^2$, $a > 1$ and $Q(0, -1)$.

7. If $a^2 = 2$, then the maximum length of PQ is

(a) $\frac{1}{\sqrt{2}}$ (b) 2 (c) $\frac{3}{\sqrt{2}}$ (d) $\frac{4}{\sqrt{3}}$

8. If $a^2 = 5$, then the maximum length of PQ is

(a) 2 (b) $\frac{3}{\sqrt{2}}$ (c) $\frac{4}{\sqrt{3}}$ (d) $\frac{5}{2}$

INTEGER MATCH

9. Let N be the number of 5-letter words using the letters of the word CALCULUS. The sum of the digits of N is

MATCHING LIST

10. Let $f(x) = \sin 2x \sin\left(\frac{\pi}{2}\cos x\right)$ and $g(x) = \frac{f(x)}{2x-\pi}$.

	Column-I		Column-II
P.	$\int_0^\pi f(x)dx =$	1.	$\frac{8}{\pi}$
Q.	$\int_0^\pi g(x)dx =$	2.	$\frac{8}{\pi^2}$
R.	$\int_0^\pi xg(x)dx =$	3.	0
S.	$\int_0^\pi x^2g(x)dx =$	4.	$\frac{16}{\pi^2}$

	P	Q	R	S
(a)	1	2	3	4
(b)	2	1	4	3
(c)	4	3	2	1
(d)	3	2	1	4

See Solution set of Maths Musing 145 on page no. 86

Prof. Ramanaiah is the author of MTG JEE(Main & Advanced) Mathematics series

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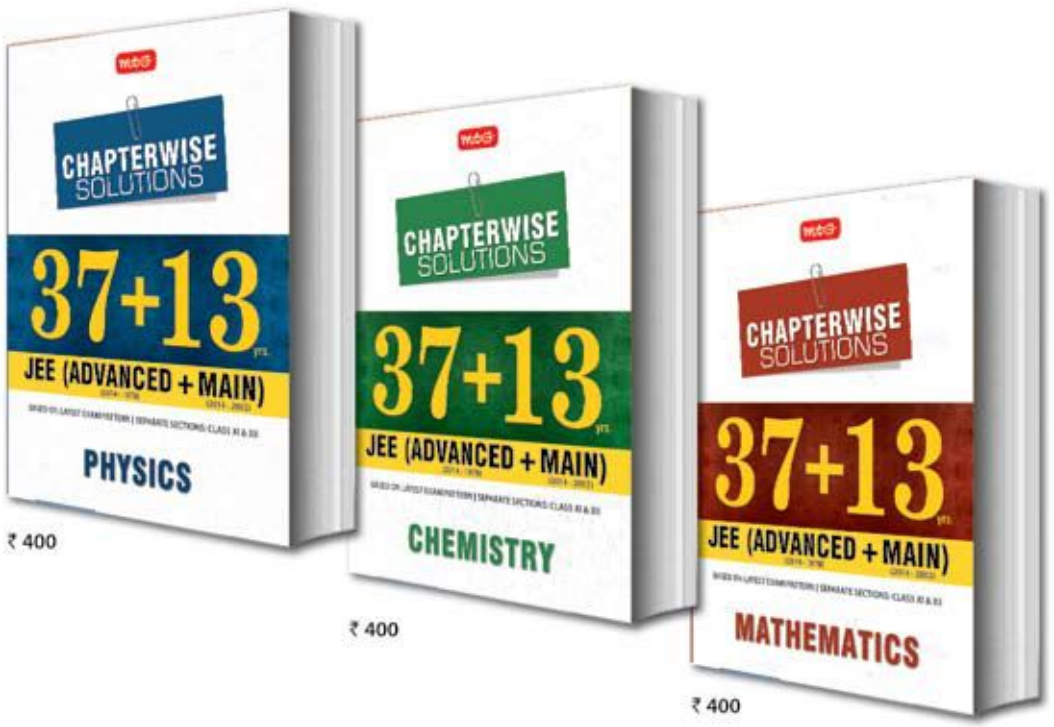

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MOCK TEST PAPER 2015

JEE Main

Exam on 4th April 2015

* ALOK KUMAR, B.Tech, IIT Kanpur

1. Set 'A' has 10 elements. A subset P of A is selected at random. After inspecting the elements, the elements are replaced into 'A' and another subset Q is selected at random. If the probability that Q has exactly one element more, then P is $\frac{{}^{m}C_{r}}{2^{k}}$ where

(a) $m = 10, r = 5, k = 10$
(b) $m = 10, r = 6, k = 20$
(c) $m = 20, r = 9, k = 20$
(d) $m = 20, r = 10, k = 10$

2. A and B are two events such that $P(\bar{A}) = 0.3$, $P(B) = 0.4$ and $P(A \cap \bar{B}) = 0.5$. Then $P(B/A \cup \bar{B}) =$

(a) $\frac{1}{4}$ (b) $\frac{1}{3}$ (c) $\frac{4}{5}$ (d) $\frac{1}{5}$

3. The locus of the mid point of the line segment joining the focus to a moving point on the parabola $y^2 = 4ax$ is another parabola with directrix

(a) $x = -a$ (b) $x = -\frac{a}{2}$

(c) $x = 0$ (d) $x = \frac{a}{2}$

4. Let $S(3, 4)$ and $S'(9, 12)$ be two foci of an ellipse. If the coordinates of the foot of the perpendicular from focus S to a tangent to the ellipse is $(1, -4)$, then the eccentricity of the ellipse is

(a) 4/5 (b) 5/7 (c) 7/13 (d) 5/13

5. PQ is a double ordinate of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ such that OPQ is an equilateral triangle, O being the centre of the hyperbola, then the eccentricity e of the hyperbola satisfies

(a) $1 < e < \frac{2}{\sqrt{3}}$ (b) $e = \frac{2}{\sqrt{3}}$

(c) $e = \frac{\sqrt{3}}{2}$ (d) $e > \frac{2}{\sqrt{3}}$

6. For the hyperbola $\frac{x^2}{\cos^2\alpha} - \frac{y^2}{\sin^2\alpha} = 1$, which of the following remains constant with change in α

(a) abscissae of vertices
(b) abscissae of foci
(c) eccentricity
(d) directrix

7. The exponent of 12 in 100! is

(a) 48 (b) 49
(c) 96 (d) none of these

8. How many different 9 digit numbers can be formed from the number 223355888 by rearranging its digits so that the odd digits occupy even positions

(a) 16 (b) 36 (c) 60 (d) 180

9. Number of points having position vector $a\hat{i} + b\hat{j} + c\hat{k}$ where $a, b, c \in \{1, 2, 3, 4, 5\}$ such that $2^a + 3^b + 5^c$ is divisible by 4 is

(a) 70 (b) 140 (c) 210 (d) 250

10. Given that n is odd, the no. of ways in which three numbers in A.P. can be selected from 1, 2, 3, 4, …, n is

(a) $\frac{(n-1)^2}{2}$ (b) $\frac{(n+1)^2}{4}$

(c) $\frac{(n+1)^2}{2}$ (d) $\frac{(n-1)^2}{4}$

* Alok Kumar is a winner of INDIAN NATIONAL MATHEMATICS OLYMPIAD (INMO-91). He trains IIT and Olympiad aspirants.

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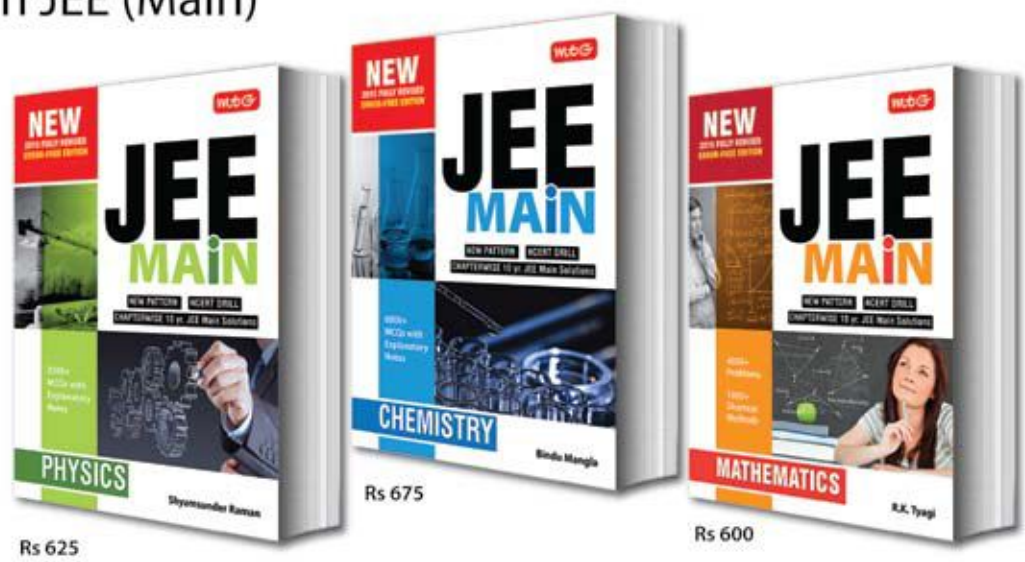

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11. A die is rolled three times, the probability of getting large number than the previous number is

(a) 1/54 (b) 5/54

(c) 5/108 (d) 13/108

12. If a, b, c are three natural numbers in A.P. and $a + b + c = 21$, then the possible number of values of the ordered triplet (a, b, c) is

(a) 15 (b) 14 (c) 13 (d) 17

13. If $f(x) = ax^3 + bx^2 + cx + d$, ($a, b, c, d$ are rationals) and roots of $f(x) = 0$ are eccentricities of a parabola and a rectangular hyperbola, then $a + b + c + d$ equals

(a) -1 (b) 0

(c) 1 (d) data inadequate

14. 10 apples are distributed at random among 6 persons. The probability that at least one of them will receive none is

(a) $\frac{6}{143}$ (b) $\frac{{}^{14}C_4}{{}^{15}C_5}$ (c) $\frac{137}{143}$ (d) $\frac{143}{137}$

15. A bag contains m white and m red balls. Pairs of balls are drawn without replacement until the bag is empty. The probability that each pair consists of one white and one red ball is

(a) $\frac{2^{m-1}}{{}^{2m}C_m}$ (b) $\frac{2^m}{{}^{2m}C_m}$

(c) $\frac{2^{m+1}}{{}^{2m}C_m}$ (d) $\frac{2^{m/2}}{{}^{2m}C_m}$

16. $\int \frac{(x^2+1)}{(x^4-x^2+1)\cot^{-1}\left(x-\frac{1}{x}\right)}dx$ is equal to

(a) $-\ln\left|\cot^{-1}\left(x-\frac{1}{x}\right)\right|+C$

(b) $\ln\left|\cot^{-1}\left(x-\frac{1}{x}\right)\right|+C$

(c) $\ln\left|x^2\cot^{-1}\left(x-\frac{1}{x}\right)\right|+C$

(d) $x^2\ln\left|\cot^{-1}\left(x-\frac{1}{x}\right)\right|+C$

17. A coin is tossed $(m + n)$ times $(m > n)$. The probability that atleast 'm' consecutive heads will appear is

(a) $\frac{n}{2^{m+1}}$ (b) $\frac{n+1}{2^{m+1}}$ (c) $\frac{n+2}{2^{m+1}}$ (d) $\frac{n-1}{2^{m+1}}$

18. If the sum of the ordinate and the abscissa of a point $P(x, y)$ is $2n$ (x, y are natural numbers), then the probability that P does not lie on the line $y = x$ is

(a) $\frac{n-1}{n+3}$ (b) $\frac{{}^{2n}C_n}{2^{2n}}$

(c) $\frac{2n-2}{2n-1}$ (d) $\frac{2n+1}{2n+3}$

19. The number of points at which the function $f(x) = \max\{2 - x, 3, 2 + x\}$ cannot be differentiable is

(a) 2 (b) 3 (c) 0 (d) 1

20. Two players A and B play a match which consists of a series of games (independent). Whoever first wins two games not necessarily consecutive, wins the match. The probability of A's winning, drawing or losing a game against B are $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$ respectively. It is known that A won the match at the end of 11^{th} game, the probability that B wins only one game is

(a) 3/11 (b) 8/11

(c) 9/11 (d) 10/11

21. If $x^2 - x + a - 3 < 0$ for at least one negative value of x, then complete set of values of a is

(a) $(-\infty, 4)$ (b) $(-\infty, 2)$

(c) $(-\infty, 3)$ (d) $(-\infty, 1)$

22. Let $f(x) = \int e^x(x-1)(x-2)dx$, then f decreases in the interval

(a) $(-\infty, -2)$ (b) $(-2, -1)$

(c) $(1, 2)$ (d) $(2, \infty)$

23. If $I = \int \frac{\sin 2x}{(3+4\cos x)^3}dx$, then I is equal to

(a) $\frac{3\cos x+8}{(3+4\cos x)^2}+C$ (b) $\frac{3+8\cos x}{16(3+4\cos x)^2}+C$

(c) $\frac{3+\cos x}{(3+4\cos x)^2}+C$ (d) $\frac{3-8\cos x}{16(3+4\cos x)^2}+C$

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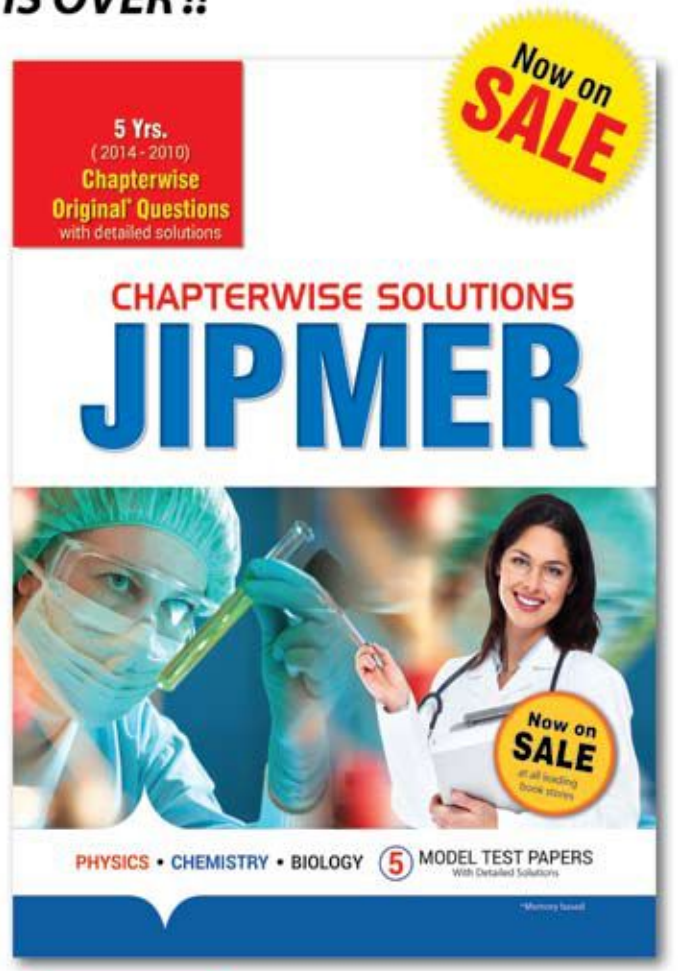

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24. Let $f(x) = \dfrac{x}{(1+x^n)^{1/n}}$ for $n \geq 2$ and $g(x) = \underbrace{(f \circ f \circ .. \circ f)}_{f \text{ occurs } n \text{ times}}(x)$, then $\int x^{n-2} g(x)dx$ equals

(a) $\dfrac{1}{n(n-1)}(1+nx^n)^{1-\frac{1}{n}} + K$

(b) $\dfrac{1}{n-1}(1+nx^n)^{1-\frac{1}{n}} + K$

(c) $\dfrac{1}{n(n+1)}(1+nx^n)^{1+\frac{1}{n}} + K$

(d) $\dfrac{1}{n+1}(1+nx^n)^{1+\frac{1}{n}} + K$

25. Find the number of different ways in which 13 distinct objects can be divided into two groups of 5 and 8

(a) 1287 (b) 1286 (c) 1280 (d) 1387

26. The number of ways of a mixed double game can be arranged from amongst 9 couples if no husband and wife play in the same game is

(a) 756 (b) 1512 (c) 3024 (d) 3000

27. The letters of the word MIRROR are arranged in all possible ways these words are written as in a dictionary, then the rank of word MIRROR will be

(a) 23 (b) 24 (c) 25 (d) 26

28. The number of six digit numbers in which digits are in ascending order

(a) 48 (b) 84 (c) 120 (d) 126

29. Consider a branch of the hyperbola $x^2 - 2y^2 - 2\sqrt{2}x - 4\sqrt{2}y - 6 = 0$ with the vertex at the point A, let B be one end of its latusrectum. If C is the focus of the hyperbola nearest to the point A, then the area of the triangle ABC is

(a) $1 - \sqrt{\dfrac{2}{3}}$ (b) $\sqrt{\dfrac{3}{2}} - 1$

(c) $1 + \sqrt{\dfrac{2}{3}}$ (d) $\sqrt{\dfrac{3}{2}} + 1$

30. A bag contains $(n + 1)$ coins. It is known that one of these coins has a head on both sides while the other coins are fair. One coin is selected at random and tossed. If head turns up, then the probability that the selected coin was fair, is

(a) $\dfrac{2}{n+2}$ (b) $\dfrac{n}{n+2}$

(c) $\dfrac{2n}{n+2}$ (d) none of these

31. A man is known to speak the truth 3 out of 4 times. He throws a die and reports that it is a six; then the probability that it is actually a six.

(a) $\dfrac{1}{8}$ (b) $\dfrac{1}{4}$ (c) $\dfrac{3}{8}$ (d) $\dfrac{1}{2}$

32. If 'x' follows a binomial distribution with parameters $n = 8$ and $p = \dfrac{1}{2}$, then $P(|x - 4| \leq 2)$ is equal to

(a) $\dfrac{121}{128}$ (b) $\dfrac{119}{128}$ (c) $\dfrac{117}{128}$ (d) $\dfrac{115}{128}$

33. Let $d_1, d_2, \ldots\ldots, d_k$ be all the divisors of a positive integer n including 1 and n. Suppose $d_1 + d_2 + \ldots + d_k = 72$.

Then the value of $\dfrac{1}{d_1} + \dfrac{1}{d_2} + \ldots\ldots + \dfrac{1}{d_k}$ is

(a) $\dfrac{k^2}{72}$ (b) $\dfrac{72}{k}$

(c) $\dfrac{72}{n}$

(d) cannot be computed from the given information

34. There are 10 stations on a circular path. A train has to stop at 3 stations such that no two stations are adjacent. The number of such selections must be

(a) 50 (b) 84

(c) 126 (d) None of these

35. Let n and k be positive integers such that $n \geq \dfrac{k(k+1)}{2}$. The number of solution $(x_1, x_2, \ldots, x_k)$, $x_1 \geq 1$, $x_2 \geq 2$, …, $x_k \geq k$, all integers, satisfying $x_1 + x_2 + \ldots + x_k = n$, is $\left(\text{where } m = \dfrac{2n - k^2 + k - 2}{2}\right)$

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(a) ${}^{m}C_{k}$ (b) ${}^{m-1}C_{k}$

(c) ${}^{m}C_{k-1}$ (d) Zero

36. An n-digit number is a positive integer with exactly n-digits. Nine hundred distinct n-digit numbers are to be formed by using the digit 2, 5 and 7 only. The smallest value of n for which this is possible is

(a) 6 (b) 7 (c) 8 (d) 9

SOLUTIONS

1. (c) : Total sample points = $2^n.2^n$ if A contains n elements.

No. of favourable cases = $C_0C_1 + C_1C_2 + C_2C_3 + ... + C_{n-1}C_n = (2n)C_{n-1}$

2. (a) : $P(\overline{A}) = 0.3, P(B) = 0.4, P(A \cap \overline{B}) = 0.5$

$\therefore \quad P(A) = 0.7,\ P(\overline{B}) = 0.6$ and $P(A \cap \overline{B}) = 0.5$

$P(A) - P(A \cap B) = 0.5 \Rightarrow P(A \cap B) = 0.7 - (0.5) = 0.2$

$\therefore \quad P(A \cup \overline{B}) = P(A) + P(\overline{B}) - P(A \cap \overline{B}) = 0.8$

$$P(B/A \cup \overline{B}) = \frac{P(B \cap (A \cup \overline{B}))}{P(A \cup \overline{B})} = \frac{0.2}{0.8} = \frac{1}{4}$$

3. (c) : $P(at^2, 2at)$, $S = (a, 0)$,

Coordinates of midpoint of SP are given by

$$x = \frac{a(t^2+1)}{2}, y = \frac{2at}{2}$$

Eliminating t, we get the locus of the mid point as

$y^2 = 2ax - a^2$ or $y^2 = 2a(x - a/2)$ ...(1)

which is a parabola of the form $y^2 = 4AX$ ...(2)

where $Y = y$, $X = x - a/2$, $A = a/2$

Equation of the directrix of (2) is $X = -A$

So, equation of the directrix of (1) is

$x - a/2 = -a/2$

$\Rightarrow \quad x = 0$

4. (d) : $SS' = 2ae \quad \Rightarrow \quad ae = 5$

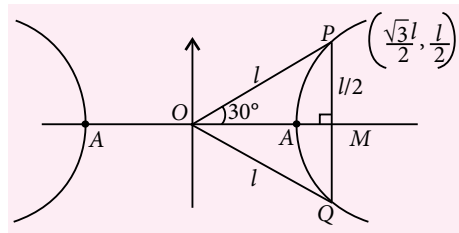
Centre = (6, 8)

Equation of auxiliary circle is $(x - 6)^2 + (y - 8)^2 = a^2$

(1, –4) lies on auxiliary circle.

$$\Rightarrow \quad a = 13,\ e = \frac{5}{13}$$

5. (d) : P lies on $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$



$$\Rightarrow \quad \frac{3l^2}{4a^2} - \frac{l^2}{4b^2} = 1 \Rightarrow \quad l = 2a\sqrt{\frac{e^2-1}{3e^2-4}}$$

$l \in R, e > 1$ and $e > \dfrac{2}{\sqrt{3}}$

6. (b) : x coordinates of foci = $(\pm ae, 0) = (\pm 1, 0)$

$$e^2 = \frac{1}{\cos^2\alpha} \Rightarrow e\cos\alpha = \pm 1$$

Abscissae of foci remains constant.

7. (a) : $12 = 2^2 \times 3 = 2 \times 2 \times 3$

$100! = 2^a \times 3^b \times 5^c \times .....$

$a = E_2(100!) = 97$

$b = E_3(100!) = 48$

$100! = 2^{97} \times 3^{48} \times 5^c \times ...$

$= (2^2 \times 3)^{48} \times 2 \times 5^c \times ....$

$= 12^{48} \times 2 \times 5^c \times ....$

$E_{12}(100!) = 48$

8. (c) : Odd digits 3, 3, 5, 5

Even places = 4

Odd digits occupy even places in $\dfrac{4!}{2!2!}$ ways.

Remaining 5 places can be filled in $\dfrac{5!}{2!3!}$ ways.

$$\text{Required no. of 9 digit numbers} = \frac{4!}{2!2!} \times \frac{5!}{2!3!} = 60$$

9. (a) : $4\lambda = 2^a + 3^b + 5^c = 2^a + (4-1)^b + (4+1)^c$

$= 2^a + 4K + (-1)^b + 1^c$

(i) $a = 1$, b = even, c = any number

(ii) $a \neq 1$, b = odd, c = any number

Required no. of ways = $1 \times 2 \times 5 + 4 \times 3 \times 5 = 70$

10. (d) : $n = 2M + 1$

Let, $M + 1$ is odd, M is even.

a, b, c are in A.P

$\therefore \quad 2b = a + c =$ even

Required no. of ways =

$$^{M}C_2 + {}^{M+1}C_2 = M^2 = \left(\frac{n-1}{2}\right)^2$$

11. (b) : If the 2nd number is $i(i > 1)$ the no. of favourable ways = $(i - 1) \times (6 - i)$

$n(E)$ = total no. of favourable ways

$$= \sum_{i=1}^{6} (i-1)\times(6-i) = 1\times4+2\times3+3\times2+4\times1 = 20$$

Required probability $= \frac{20}{216} = \frac{5}{54}$

12. (c) : $a + a + d + a + 2d = 21$ or $a + d = 7$

$\therefore \quad a + c = 14$ and $b = 7$.

The number of positive integral solutions of $a + c = 14$ is 13.

13. (b) : Roots of $f(x)$ are $1, \sqrt{2}, -\sqrt{2}$

$$ax^3 + bx^2 + cx + d = (x-1)(x-\sqrt{2})(x+\sqrt{2})$$

$$= (x-1)(x^2-2) = x^3 - x^2 - 2x + 2$$

$a = 1, b = -1, c = -2, d = 2 \implies a + b + c + d = 0$

14. (c) : The required probability = 1 – probability of each receiving at least one $= 1 - \frac{n(E)}{n(S)}$.

Now, the number of integral solutions of

$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 10$

such that $x_1 \geq 1, x_2 \geq 1, \ldots\ldots, x_6 \geq 1$ gives $n(E)$ and the number of integral solutions of $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 10$ such that $x_1 \geq 0, x_2 \geq 0, \ldots, x_6 \geq 0$ gives $n(S)$

$\therefore$ The required probability

$$= 1 - \frac{^{10-1}C_{6-1}}{^{10+6-1}C_{6-1}} = 1 - \frac{^{9}C_5}{^{15}C_5} = \frac{137}{143}$$

15. (b) :

$$P(E) = \frac{^{m}C_1 . {}^{m}C_1}{^{2m}C_2} \cdot \frac{^{m-1}C_1 . {}^{m-1}C_1}{^{2m-2}C_2} \ldots\ldots \frac{1}{^{2}C_2} = \frac{2^m}{^{2m}C_m}$$

16. (a) : $\frac{d}{dx}\left(-\ln\left|\cot^{-1}\left(x - \frac{1}{x}\right)\right| + C\right)$

$$= \frac{-1}{\cot^{-1}\left(x-\frac{1}{x}\right)} \cdot \left(\frac{-1}{1+\left(x-\frac{1}{x}\right)^2}\right) \cdot \left(1 + \frac{1}{x^2}\right)$$

17. (c)

18. (c) : No. of solutions of $x + y = 2n$ $(x, y \in N)$ is $2n - 1$, in which one is $n + n$, so required probability is $\frac{2n-2}{2n-1}$.

19. (a) : At $x = -1$, L.H.D. $\neq$ R.H.D. and at $x = 1$, L.H.D. $\neq$ R.H.D.

20. (c) : $E_1 = A$ wins the match at the end of 11th game

$E_2 = B$ wins exactly one game

$\therefore \quad P\left(\frac{E_2}{E_1}\right)$

$$= \frac{^{10}C_2 \cdot \left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{3}\right)^8 \cdot \frac{1}{6}(2!)}{^{10}C_1\left(\frac{1}{2}\right)^2\left(\frac{1}{3}\right)^9 + {}^{10}C_2\left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{3}\right)^8 \cdot \left(\frac{1}{6}\right)2!} = \frac{9}{11}$$

21. (c) : For real roots, $D \geq 0 \implies a \leq \frac{13}{4}$

Both roots will be non-negative if $D \geq 0$, $a - 3 \geq 0$;

$\implies a \in \left[3, \frac{13}{4}\right]$

$\therefore$ For at least one negative root if

$$a \in \left(-\infty, \frac{13}{4}\right) - \left[3, \frac{13}{4}\right]$$

$\implies a \in (-\infty, 3)$

22. (c) : $f'(x) = e^x (x - 1) (x - 2)$, $f'(x) < 0$

$\implies x \in (1, 2)$

23. (b) : Write $I = \int \frac{2\sin x \cos x}{(3 + 4\cos x)^3} dx$ and

put $3 + 4\cos x = t$ so that $-4 \sin x\, dx = dt$ and

$$I = \frac{-1}{8}\int \frac{(t-3)}{t^3} dt = \frac{1}{8}\left(\frac{1}{t} - \frac{3}{2}\frac{1}{t^2}\right) + C$$

$$= \frac{8\cos x + 3}{16(3 + 4\cos x)^2} + C$$

24. (a) : $f(x) = \frac{x}{(1 + x^n)^{1/n}}$

$$fof(x) = \frac{x}{(1+2x^n)^{1/n}} \implies \underbrace{fofof\ldots of}_{n \text{ times}}(x) = \frac{x}{(1+nx^n)^{1/n}}$$

$$g(x) = \frac{x}{(1+nx^n)^{1/n}}$$

$$\int x^{n-2} g(x) dx = \frac{1}{n(n-1)}[1 + nx^n]^{1-1/n} + K$$

Contd. on Page No. 85

Contd. from Page No. 17

25. (a) : Use $\frac{n!}{|p|q}$ division of 13 objects into two unequal groups of 5 & 8.

26. (b) : 9 Couples → 9 Men, 9 Women

First selecting two men, removing corresponding wives, from remaining 7 women two can be selected in ${}^7C_2$ ways

No. of ways = ${}^9C_2 \times {}^7C_2 \times 2$

27. (a) : According to dictionary : I, M, O, R, R, R

Words begin with $I \rightarrow \frac{5!}{3!} = 20$

Word begin with MIO → 1

Word begin with MIRO → 1

Word begin with MIRROR → 1

Hence, rank of word MIRROR is 23

28. (b) : It is equivalent to

$$^9C_6 = {}^9C_3 = \frac{9\times8\times7}{6} = 84$$

29. (b) : $x^2 - 2y^2 - 2\sqrt{2}x - 4\sqrt{2}y - 6 = 0$

$(x-\sqrt{2})^2 - 2(y+\sqrt{2})^2 = 4$

30. (b) : Let X = {selected coin is fair}

H = {head turns up}

Thus $P(H) = P(X)\ P(H/X) + P(X')\ P(H/X')$

$$= \frac{n}{n+1}\times\frac{1}{2} + \frac{1}{n+1}\times1 = \frac{n+2}{2(n+1)}$$

$$P(X/H) = \frac{P(X)P\left(H/X\right)}{P\left(H\right)} = \frac{\frac{n}{2(n+1)}}{\frac{n+2}{2(n+1)}} = \frac{n}{n+2}$$

31. (c) : A = The event that man reports occurrence of 6.

A_1= The event of occurrence of 6 when a die is thrown

A_2 = The event of non-occurrence of 6 when a die is thrown

$$P(A_1) = \frac{1}{6}, P(A_2) = \frac{5}{6}$$

$$P\left(\frac{A_1}{A}\right) = \frac{P(A_1)P\left(A/A_1\right)}{P(A_1)P\left(A/A_1\right) + P(A_2)P\left(A/A_2\right)}$$

$$= \frac{\frac{1}{6}\times\frac{3}{4}}{\frac{1}{6}\times\frac{3}{4} + \frac{5}{6}\times\frac{1}{4}} = \frac{3}{8}$$

32. (b) : $p = \frac{1}{2}, n = 8 \Rightarrow q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$

$\therefore$ B.D. $= \left(\frac{1}{2} + \frac{1}{2}\right)^8$

$P(|x-4| \le 2) = P(x=2) + P(x=3) + P(x=4) + P(x=5) + P(x=6)$

33. (c) : $\frac{1}{d_1} + \frac{1}{d_2} + ... + \frac{1}{d_k}$

$$= \frac{1}{n}\left[\frac{n}{d_1} + \frac{n}{d_2} + \frac{n}{d_3} + ... + \frac{n}{d_k}\right]$$

Now $\frac{n}{d_1}, \frac{n}{d_2}, ...$ will also be divisor of the number, *i.e.*,

$\frac{n}{d_j} = d_l$ for same j and l.

$$\Rightarrow \frac{1}{d_1} + \frac{1}{d_2} + ... + \frac{1}{d_k} = \frac{1}{n}[d_1 + d_2 + ...] = \frac{72}{n}$$

34. (a) : Total selections = ${}^{10}C_3 = 120$

Number of selections in which 3 stations are adjacent = 10

Number of selections in which 2 stations are adjacent = 6

But there are 10 such pairs.

$\Rightarrow$ Total invalid selections = $10 + 6 \times 10 = 70$

35. (c) : Put $y_1 = x_1 - 1, y_2 = x_2 - 2, \ldots, y_k = x_k - k$

On adding, $y_1 + y_2 + ... + y_k = n - \frac{k(k+1)}{2}$ etc.

36. (b) : We must have $3^n > 900$.

The least n satisfying this is 7.

PRACTICE PAPER 2015

JEE (Main & Advanced) and other PETs

PAPER-1

SECTION-I

MULTIPLE CORRECT CHOICE TYPE

This section contains 10 multiple choice questions. Each question has four choices (a), (b), (c) and (d) out of which ONE or MORE may be correct. [Correct answer 3 marks and wrong answer no negative mark]

1. If e_1 and e_2 are the eccentricities of the conic sections $16x^2 + 9y^2 = 144$ and $9x^2 - 16y^2 = 144$, then

(a) $e_1^2 + e_2^2 = 3$ (b) $e_1^2 + e_2^2 > 3$

(c) $e_1^2 + e_2^2 < 3$ (d) $e_1^2 - e_2^2 < 0$

2. The equation(s) to the tangent(s) to the conic $x^2 + 4xy + 3y^2 - 5x - 6y + 3 = 0$, which are parallel to $x + 4y = 0$ are

(a) $x + 4y - 1 = 0$ (b) $x + 4y - 3 = 0$

(c) $x + 4y - 5 = 0$ (d) $x + 4y - 8 = 0$

3. Consider the parabola $y^2 = 4ax$ and $x^2 = 4by$. The straight line $b^{1/3}y + a^{1/3}x + a^{2/3}b^{2/3} = 0$

(a) touches $y^2 = 4ax$

(b) touches $x^2 = 4by$

(c) intersects both parabolas in real points

(d) touches first and intersect other

4. The coordinates of a point on the parabola $y^2 = 8x$ whose distance from the circle $x^2 + (y + 6)^2 = 1$ is minimum is

(a) (2, 4) (b) (2, – 4)

(c) (18, –12) (d) (8, 8)

5. The angle between the asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

(a) $\cos^{-1}\left(\frac{1}{e}\right)$ (b) $2\cos^{-1}\left(\frac{1}{e}\right)$

(c) $\sin^{-1}\left(\frac{1}{e}\right)$ (d) none of these

6. The circle $x^2 + y^2 + 4x - 6y + 3 = 0$ is one of the circles of a coaxial system of circles having as radical axis the line $2x - 4y + 1 = 0$. Then the equation of the circle of the system which touches the line $x + 3y - 2 = 0$ is

(a) $x^2 + y^2 + 2x - 2y + 2 = 0$

(b) $x^2 + y^2 + 2x + 6y = 0$

(c) $x^2 + y^2 - 2x + 6y = 0$

(d) $x^2 + y^2 + 2x - 6y = 0$

7. If a circle of constant radius $3k$ passes through the origin and meets the axis at A and B, the locus of the centroid of ΔOAB is

(a) $x^2 + y^2 = k^2$ (b) $x^2 + y^2 = 2k^2$

(c) $x^2 + y^2 = 3k^2$ (d) none of these

8. $\frac{x^2}{P^2 - P - 6} + \frac{y^2}{P^2 - 6P + 5} = 1$ will represent the ellipse if P lies in the interval

(a) $(-\infty, -2)$ (b) $(1, \infty)$

(c) $(3, \infty)$ (d) $(5, \infty)$

9. If the eccentric angles of the extremities of a focal chord of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are α and β, then

(a) $e = \frac{\cos\alpha + \cos\beta}{\cos\alpha + \beta}$

By : **Vidyalankar Institute**, Pearl Centre, Senapati Bapat Marg, Dadar (W), Mumbai - 28. Tel.: (022) 24306367

(b) $e=\dfrac{\sin\alpha+\sin\beta}{\sin\alpha+\beta}$

(c) $\cos\left(\dfrac{\alpha-\beta}{2}\right)=e\cos\left(\dfrac{\alpha+\beta}{2}\right)$

(d) $\tan\dfrac{\alpha}{2}\cdot\tan\dfrac{\beta}{2}=\dfrac{e-1}{e+1}$

10. If b and c are the lengths of the segments of any focal chord of a parabola $y^2 = 4ax$, then the length of the semi latus rectum is

(a) $\dfrac{b+c}{2}$ (b) $\dfrac{bc}{b+c}$

(c) $\dfrac{2bc}{b+c}$ (d) $\sqrt{bc}$

SECTION-II

ONE INTEGER VALUE CORRECT TYPE

This section contains 10 questions. Each question, when worked out will result in one integer from 0 to 9 (both inclusive). [Correct answer 3 marks & wrong answer no negative mark]

11. The locus of the centre of the circle for which one end of diameter is (3, 3) while the other end lies on the line $x + y = 4$ is $x + y = k$, then k equals

12. The greatest distance of the point (10, 7) from the circle $x^2 + y^2 - 4x - 2y - 20 = 0$ is 5α, then α equals

13. Angle between the tangents drawn from (1, 4) to the parabola $y^2 = 4x$ is π/m, where m equals

14. If the straight line $y = 2x + c$ is a tangent to the ellipse $\dfrac{x^2}{8}+\dfrac{y^2}{4}=1$, then $|c|$ equals

15. The foci of the ellipse $\dfrac{x^2}{16}+\dfrac{y^2}{b^2}=1$ and $\dfrac{x^2}{144}-\dfrac{y^2}{81}=\dfrac{1}{25}$ the hyperbola coincide, then value of b^2 equals

16. If the angle between the two lines represented by $2x^2 + 5xy + 3y^2 + 7y + 4 = 0$ is $\tan^{-1}(m)$, then the value of $10m$ must be

17. From a point, common tangents are drawn to the circle $x^2 + y^2 = 8$ and parabola $y^2 = 16x$. If the area of the quadrilateral formed by the common tangents, the chord of contact of the circle and the chord of contact of the parabola is 10 k, then find k.

18. The line $x + y = a$ meets the x-axis at A and y-axis at B. ΔAMN is inscribed in the ΔOAB, O being the origin, with right angled at N. M and N lie respectively on OB and AB. If ratio of the $\dfrac{\text{Area }(\Delta AMN)}{\text{Area }(\Delta OAB)}=\dfrac{3}{8}$, then find the value of $\dfrac{AN}{BN}$.

19. If $A(0, 0)$, $B(4, 2)$ and $C(6, 0)$ are the vertices of a triangle ABC and BD is its altitude. The line through D parallel to the side AB intersects the side BC at a point E. Find the product of areas of ΔABC and ΔBDE.

20. Find the number of integral values of λ if $(\lambda, 2)$ is an interior point of ΔABC formed by $x + y = 4$, $3x - 7y = 8$, $4x - y = 31$.

PAPER-2

SECTION-I

SINGLE CORRECT OPTION

This section contains 10 multiple choice questions. Each question has four choices (a), (b), (c) and (d) for its answer, out of which ONLY ONE is correct. [Correct answer 3 marks and wrong answer –1 mark]

1. If $x + ky = 1$ and $x = a$ are the equations of the hypotenuse and a side of a right angled isosceles triangle, then

(a) $k = \pm 1$ (b) $k = \pm a$

(c) $k = \pm 1/a$ (d) $k = \pm 2$

2. Given two points $A \equiv (-2, 0)$ and $B \equiv (0, 4)$. The co-ordinates of a point M lying on the line $x = y$ so that the perimeter of the ΔAMB is least, is

(a) (1, 1) (b) (0, 0)

(c) (2, 2) (d) (3, 3)

3. A ray of light travels along the line $2x - 3y + 5 = 0$ and strikes a plane mirror lying along the line $x + y = 2$. The equation of the straight line containing the reflected ray is

(a) $2x - 3y + 3 = 0$ (b) $3x - 2y + 3 = 0$

(c) $21x - 7y + 1 = 0$ (d) $21x + 7y - 1 = 0$

4. If the inclination of the diameter PP' of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ to the major axis is θ and PP'^2 is the A.M. of squares of major and minor axis, then $\tan\theta$ is equal to

(a) b/a (b) a/b

(c) $\pi/4$ (d) $\pi/6$

5. If a circle of radius r is concentric with ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then the common tangent is inclined to major axis at an angle

(a) $\tan^{-1}\sqrt{\frac{r^2-b^2}{a^2-r^2}}$ (b) $\tan^{-1}\sqrt{\frac{r^2-a^2}{b^2-r^2}}$

(c) $\tan^{-1}\sqrt{\frac{r^2-b^2}{r^2-a^2}}$ (d) $\tan^{-1}\sqrt{\frac{r^2-a^2}{r^2-b^2}}$

6. The diagonals of a square are along the pair of lines whose equation is $2x^2 - 3xy - 2y^2 = 0$. If (2, 1) is a vertex of the square, then another vertex consecutive can be

(a) (1, –2) (b) (1, 4)

(c) (2, –1) (d) (–1, –2)

7. All the chords of curve $3x^2 - y^2 - 2x + 4y = 0$ which subtend a right angle at the origin pass through

(a) (1, 2) (b) (1, –2)

(c) (2, 1) (d) (0, 0)

8. A circle passing through origin O cuts two straight lines $x - y = 0$ and $x + y = 0$ in points A and B respectively. If abscissa of A and B are roots of the equation $x^2 + ax + b = 0$, then the equation of the given circle in terms of a and b is

(a) $x^2 + y^2 + ax - by = 0$

(b) $x^2+y^2-x\sqrt{4b-a^2}+yb=0$

(c) $x^2+y^2+ax\pm y\sqrt{a^2-4b}=0$

(d) $x^2+y^2-ax\pm y\sqrt{a^2-4b}=0$

9. The locus of the mid-points of the chords of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ which are tangent to the ellipse $\frac{x^2}{p^2}+\frac{y^2}{q^2}=1$ is

(a) $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2=\frac{p^2x^2}{a^4}+\frac{q^2y^2}{b^4}$

(b) $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2=\frac{x^2}{p^2}+\frac{y^2}{q^2}$

(c) $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2=\frac{p^2}{a^2}+\frac{q^2}{b^2}$

(d) $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2=\frac{p^2}{q^2}+\frac{a^2}{b^2}$

10. If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents two lines equidistant from the origin, then value of $f^4 - g^4$ will be

(a) $c(ag^2 - bf^2)$ (b) $c(af^2 - bg^2)$

(c) $c(bf^2 - ag^2)$ (d) $-c(af^2 - bg^2)$

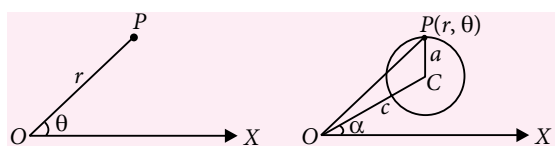
SECTION-II

PARAGRAPH TYPE

This section contains 2 paragraph. Based upon each of the paragraphs 3 multiple choice questions have to be answered. Each of these questions has four choices (a), (b), (c) and (d) out of which ONLY ONE is correct. [Correct answer 3 marks & wrong answer –1 mark]

Paragraph for Q. No. 11 to 13

Suppose OX is a fixed line on which O is a fixed point. Suppose P is a point such that $OP = r$ and anticlockwise angle $XOP = \theta$, then we define (r, θ) as polar coordinates of the point P. O is called pole and line OX is called initial line.



Let the polar coordinates of the centre C of a circle be (c, α) and a be the radius of the circle. Let P be any point (r, θ) on the circle. Then, in the triangle OPC, we have

$PC^2 = OC^2 + OP^2 - 2OC \cdot OP \cos OPC$

i.e. $a^2 = c^2 + r^2 - 2\,c \cdot r \cdot \cos(\theta - \alpha)$

The equation of the circle is therefore,

$r^2 - 2 \cdot cr \cos(\theta - \alpha) + c^2 - a^2 = 0$

We can derive following corollaries :

1. If the centre of the circle lies on the pole, then $c = 0$ and the equation of the circle becomes $r^2 = a^2$.
2. If the circle passes through the pole, then $c = a$ and the equation of the circle becomes $r = 2a\cos(\theta - \alpha)$.

11. The straight line $\frac{1}{r} = A\cos\theta + B\sin\theta$ touches the circle at $r = 2a\cos\theta$, if

(a) $a^2B^2 + 2aA = 1$

(b) $a^2A^2 + 2aA + a^2B^2 = 1$

(c) $a^2A^2 + 2AB = 1$

(d) $a^2A^2 + 2aA = 1$

12. If a circle passes through the point (r_1, θ_1) and touches the initial line at a distance c from the pole, then its polar equation is $\frac{r^2 - 2cr\cos\theta + c^2}{r\sin\theta} = \lambda$, where $\lambda =$

(a) $r_1\sin\theta_1$

(b) $r_1^2 - 2cr_1\cos\theta + c^2$

(c) $\frac{r_1^2 - 2cr_1\cos\theta_1 + c^2}{r_1\sin\theta_1}$

(d) $\frac{r_1^2 + 2cr_1\cos\theta_1 + c^2}{r_1\sin\theta_1}$

13. O is a fixed point, P is any point on a given circle. OP is joined and on it is taken a point Q such that $OP \cdot OQ$ is a constant quantity k, then the locus of Q is a circle, whose radius is ($OC = c$ and radius of circle $= a$, C being centre)

(a) $\frac{k^2a^2}{|c^2 - a^2|}$ (b) $\frac{\sqrt{1-k^2}\,c}{|c^2 - a^2|}$

(c) $\left|\frac{ka}{c^2 - a^2}\right|$ (d) $\left|\frac{kc}{c^2 - a^2}\right|$

Paragraph for Q. No. 14 to 16

Suppose that an ellipse and a circle are respectively given by the equation

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ... (1)

and $x^2 + y^2 + 2gx + 2fy + c = 0$ ... (2)

The equation,

$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) + \lambda(x^2 + y^2 + 2gx + 2fy + c) = 0$ ... (3)

represents a curve which passes through the common points of the ellipse (1) and the circle (2).

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We can choose λ so that the equation (3) represents a pair of straight lines. In general, we get three values of λ, indicating three pair of straight lines that can be drawn through the points. Also when (3) represents a pair of straight lines they are parallel to the lines $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\lambda(x^2+y^2)=0$, which represents a pair of lines equally inclined to the axes the term containing xy is absent). Hence two straight lines through the points of intersection of an ellipse and any circle make equal angles with the axes. Above description can be applied identically for a hyperbola and a circle.

14. If α, β, γ and δ be eccentric angles of the four concyclic points of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then $\alpha+\beta+\gamma+\delta=$

(a) $(2n+1)\frac{\pi}{2}$ (b) $(2n+1)\pi$

(c) $2n\pi$ (d) $n\pi$

(n is any integer)

15. Let the eccentric angles of three points P, Q and R on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are α, $\frac{\pi}{2}+\alpha$ then $\pi+\alpha$. A circle through P, Q and R cuts the ellipse again at S, then the eccentric angle of S is

(a) $\pi-3\alpha$ (b) $\frac{3\pi}{2}-3\alpha$

(c) $\frac{\pi}{2}-3\alpha$ (d) $-\frac{\pi}{2}+3\alpha$

16. Suppose two lines are drawn through the common points of intersection of hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and circle $x^2+y^2+2gx+2fy+c=0$. If these lines are inclined at angles α and β to x-axis, then

(a) $\alpha=\beta$ (b) $\alpha+\beta=\frac{\pi}{2}$

(c) $\alpha+\beta=\pi$ (d) $\alpha+\beta=2\tan^{-1}\left(\frac{b}{a}\right)$

SECTION-III

MATCHING LIST TYPE

This section contains 4 questions, each having two matching lists. Choices for the correct combination of elements from List-I and List-II are given as options (a), (b), (c) and (d), out of which one is correct. [Correct answer 3 marks & wrong answer –1 mark]

17. Match the following:

	Column I		Column II
(P)	The radius of the circle passing through the foci of the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ and having centre at (0, 3) is	(i)	$\frac{2}{\sqrt{3}}$
(Q)	If the length of the major axis of an ellipse is three times the length of its minor axis, then its eccentricity is	(ii)	$\frac{2\sqrt{2}}{3}$
(R)	The eccentricity of hyperbola whose latus rectum is 8 and conjugate axis is half the distance between foci, is	(iii)	$\sqrt{\frac{17}{20}}$
(S)	The value of m for which $y = mx + 6$ is a tangent to the hyperbola $\frac{x^2}{100}-\frac{y^2}{49}=1$ is	(iv)	4

	P	Q	R	S
(a)	(iv)	(i)	(ii)	(iii)
(b)	(iv)	(ii)	(i)	(iii)
(c)	(iv)	(i)	(iii)	(ii)
(d)	(i)	(ii)	(iii)	(iv)

18. Consider the general equation of second degree $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$. If this represents a pair of straight lines, map the two columns in the most accurate sense.

Column I		Column II	
(P)	If (x_1, y_1) is the point of intersection of the two lines, then $(ax_1 + hy_1)$ $(hx_1 + by_1)$ =	(i)	$\frac{c}{\sqrt{(a-b)^2+4h^2}}$
(Q)	$af^2 + bg^2 + ch^2$ =	(ii)	ab
(R)	The lines are parallel if h^2 =	(iii)	gf
(S)	Product of perpendiculars from the origin	(iv)	$abc + 2fgh$

	P	Q	R	S
(a)	(iii)	(i)	(ii)	(iv)
(b)	(ii)	(i)	(iii)	(iv)
(c)	(iii)	(iv)	(ii)	(i)
(d)	(i)	(iii)	(ii)	(iv)

19. Match the following :

Column I		Column II	
(P)	Maximum number of common normal of $y^2 = 4ax$ and $x^2 = 4by$ may be equal to	(i)	3
(Q)	Given the two ends of the latus rectum, the maximum number of parabolas that can be drawn is	(ii)	5
(R)	The number of real tangents that can be drawn to the ellipse $3x^2 + 5y^2 = 32$ and $25x^2 + 9y^2 = 450$ passing through (3, 5) is	(iii)	2
(S)	The length of the common chord of the ellipse $\frac{(x-1)^2}{9}+\frac{(y-2)^2}{4}=1$ and the circle $(x-1)^2 + (y-2)^2 = 1$ is	(iv)	0

	P	Q	R	S
(a)	(i)	(iii)	(iv)	(ii)
(b)	(i)	(ii)	(iii)	(iv)
(c)	(ii)	(i)	(iii)	(iv)
(d)	(ii)	(iii)	(i)	(iv)

20. Match the following :

Column I		Column II	
(P)	The tangent to the parabola $y^2 = 4ax$ at the point $(a, 2a)$ makes with x-axis an angle equal to	(i)	$\pi/3$
(Q)	The angle between the asymptotes of the hyperbola $27x^2 - 9y^2 = 24$ is	(ii)	$\pi/4$
(R)	If the tangent at the point $\left(4\cos\theta, \frac{16}{\sqrt{11}}\sin\theta\right)$ to the ellipse $16x^2 + 11y^2 = 256$ is also a tangent to the circle $x^2 + y^2 - 2x = 15$, then the value of θ is	(iii)	$3\pi/4$
(S)	Eccentric angle of a point on the ellipse $x^2 + 3y^2 = 6$ at a distance 2 units from the centre of the ellipse is	(iv)	$-\pi/3$

	P	Q	R	S
(a)	(ii)	(i)	(i), (iv)	(ii), (iii)
(b)	(i), (ii)	(ii)	(i), (iii)	(ii), (iv)
(c)	(ii)	(i)	(iii)	(iv)
(d)	(i), (ii)	(i)	(i), (iv)	(ii)

ANSWER KEYS

PAPER-1

1.	(c, d)	**2.**	(c, d)	**3.**	(a, b)
4.	(b)	**5.**	(b)	**6.**	(a, c)
7.	(d)	**8.**	(a, d)	**9.**	(b, c, d)
10.	(c)	**11.**	(5)	**12.**	(3)
13.	(3)	**14.**	(6)	**15.**	(7)
16.	(2)	**17.**	(6)	**18.**	(3)
19.	(8)	**20.**	(1)		

PAPER - 2

1.	(a)	**2.**	(b)	**3.**	(b)	**4.**	(a)	**5.**	(a)
6.	(a)	**7.**	(b)	**8.**	(c)	**9.**	(a)	**10.**	(c)
11.	(a)	**12.**	(c)	**13.**	(c)	**14.**	(c)	**15.**	(c)
16.	(c)	**17.**	(b)	**18.**	(c)	**19.**	(d)	**20.**	(a)

For detailed solution to the Practice Paper, visit our website. www.vidyalankar.org

Math Archives

10 Best Problems

Math Archives, as the title itself suggests, is a collection of various challenging problems related to the topics of JEE (Main & Advanced) Syllabus. This section is basically aimed at providing an extra insight and knowledge to the candidates preparing for JEE (Main & Advanced). In every issue of MT, challenging problems are offered with detailed solution. The readers' comments and suggestions regarding the problems and solutions offered are always welcome.

1. Let

$$p=\left(1+\cos\frac{\pi}{10}\right)\left(1+\cos\frac{3\pi}{10}\right)\left(1+\cos\frac{7\pi}{10}\right)\left(1+\cos\frac{9\pi}{10}\right)$$

and $q=\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\frac{5\pi}{8}\right)\left(1+\cos\frac{7\pi}{8}\right)$, then

(a) $p = q$ (b) $2p = q$
(c) $p = 2q$ (d) $p + q = 1/4$

2. Let $2x - 3y = 0$ be a given line and $P(\sin\theta, 0)$ and $Q(0, \cos\theta)$ be two points. Then P and Q lie on the same side of the given line if θ lies in the

(a) 1st quadrant (b) 2nd quadrant
(c) 3rd quadrant (d) 4th quadrant

3. If $\sin^4 x\cos 3x=\sum_{k=0}^{n} a_k\cos kx$. Then the value of n and all the a_k's

(a) 5 (b) 6
(c) 7 (d) none of these

4. Let $f : R \to R$ be a function defined by, $f(x)=\frac{2x^2-x+5}{7x^2+2x+10}$, then f is

(a) injective but not surjective.
(b) surjective but not injective.
(c) injective as well as surjective.
(d) neither injective nor surjective.

5. If α and β are two distinct roots of the equation $a\tan x + b\sec x = c$, then $\tan(\alpha + \beta)$ is equal to

(a) $\frac{ac}{a^2+c^2}$ (b) $\frac{2ac}{a^2+c^2}$
(c) $\frac{2ac}{a^2-c^2}$ (d) none of these

6. If $g(\theta)=\sin^2\theta+\sin^2\left(\theta+\frac{\pi}{3}\right)+\cos\theta.\cos\left(\theta+\frac{\pi}{3}\right)$ and $f\left(\frac{5}{4}\right)=1$, then find $(fog)(x)$.

7. If $\{x\}$ and $[x]$ denote the fractional and integral parts of a real number x respectively, then solve $2x + \{x + 1\} = 4[x + 1] - 6$.

8. Find the domain of the following functions :

(a) $\frac{\cos^{-1}x}{[x]}$, where $[\cdot]$ denotes the greatest integer function.

(b) $\frac{1}{x}+2^{\sin^{-1}x}+\frac{1}{\sqrt{x-2}}$.

9. Evaluate $\lim_{x\to 0}\frac{1-\cos x.\sqrt{\cos 2x}}{x^2}$ (without using L' Hospital Rule)

10. If $f(x)=\begin{cases}(1+|\sin x|)^{\frac{p}{|\sin x|}} & ;-\frac{\pi}{6}<x<0\\ q & ;x=0\\ e^{\frac{\tan 3x}{\tan 5x}} & ;0<x<\frac{\pi}{6}\end{cases}$

is continuous at $x = 0$, find the values of p and q.

By : **Prof. Shyam Bhushan,** Director, Narayana IIT Academy, Jamshedpur. Mob. : 09334870021

SOLUTIONS

1. **(b) :** $p=\left(\sin\frac{\pi}{10}\sin\frac{3\pi}{10}\right)^2=\frac{1}{16}$

and $q=\left(\sin\frac{\pi}{8}\sin\frac{3\pi}{8}\right)^2=\frac{1}{8}$

Hence $q = 2p$.

2. **(b) :** $L\equiv 2x-3y; L(P)\cdot L(Q)>0$

$L(\sin\theta, 0)\cdot L(0, \cos\theta) > 0$

$\sin\theta\cdot\cos\theta < 0$ or $\sin2\theta < 0$

$\therefore \frac{\pi}{2}<\theta<\pi$

3. **(c) :** $n = 7$, $a_1 = -3/16$, $a_3 = 3/8$, $a_5 = -1/4$, $a_7 = 1/16$ and $a_k = a_2 = a_4 = a_6 = 0$

4. **(d)**

5. **(c) :** $(a^2 - b^2)\tan^2x - 2ac\tan x + (c^2 - b^2) = 0$

$\therefore \tan\alpha+\tan\beta=\frac{2ac}{a^2-b^2}$

$\tan\alpha\tan\beta=\frac{c^2-b^2}{a^2-b^2}$

Hence, $\tan(\alpha+\beta)=\frac{2ac}{a^2-c^2}$.

6. $g(\theta)=\sin^2\theta+\sin^2\left(\theta+\frac{\pi}{3}\right)+\cos\theta\cdot\cos\left(\theta+\frac{\pi}{3}\right)$

$=\frac{1}{2}\left[1-\cos2\theta+1-\cos\left(2\theta+\frac{2\pi}{3}\right)\right.$

$\left.+\cos\left(2\theta+\frac{\pi}{3}\right)+\cos\frac{\pi}{3}\right]$

$=\frac{1}{2}\left[\frac{5}{2}-2\cos\left(2\theta+\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}\right)+\cos\left(2\theta+\frac{\pi}{3}\right)\right]$

$=\frac{5}{4}\forall\theta$

$\therefore$ $(fog)(x) = f[g(x)]$

$=f\left(\frac{5}{4}\right)=1$

7. $2x + \{x + 1\} = 4[x + 1] - 6$

$\Rightarrow$ $2x + x + 1 - [x] - 1 = 4[x] - 2$

$\Rightarrow$ $5[x] = 3x + 2$ ... (1)

$= 3([x] + \{x\}) + 2$

$\Rightarrow$ $3\{x\} = 2[x] - 2$ ... (2)

Now, $0 \le \{x\} < 1$

$\Rightarrow$ $0 \le 3\{x\} < 3$ $\Rightarrow$ $0 \le 2[x] - 2 < 3$

$\Rightarrow$ $2 \le 2[x] < 5$ $\Rightarrow$ $1\le[x]<\frac{5}{2}$

$\Rightarrow$ $[x] = 1, 2$

$\Rightarrow$ $[x] = 1$ $\Rightarrow$ $x = 1$ and

$[x]=2$ $\Rightarrow$ $x=\frac{8}{3}$ [$\because$ from (1)]

8. (a) $D_{\cos^{-1}x} = [-1, 1]$

$D_{[x]} = R$

$\therefore$ $D_{\frac{\cos^{-1}x}{[x]}}=[-1,1]\cap R-\{x\,|\,[x]=0\}$

$= [-1, 0) \cup \{1\}$

(b) $D_{\frac{1}{x}+2^{\sin^{-1}x}+\frac{1}{\sqrt{x-2}}}=R-\{0\}\cap[-1,1]\cap(2,\infty)$

$= \phi$

$\therefore$ $f(x)$ is not defined for any $x \in R$.

9. $\lim_{x\to0}\frac{1-\cos x\cdot\sqrt{\cos2x}}{x^2}$

$=\lim_{x\to0}\frac{1-\cos^2x\cdot\cos2x}{x^2(1+\cos x\cdot\sqrt{\cos2x})}$

$=\lim_{x\to0}\frac{1-\cos^2x(2\cos^2x-1)}{x^2(1+\cos x\cdot\sqrt{\cos2x})}$

$=-\lim_{x\to0}\frac{2\cos^4x-\cos^2x-1}{x^2(1+\cos x\sqrt{\cos2x})}$

$=\lim_{x\to0}\left(\frac{2\cos^2x+1}{1+\cos x\sqrt{\cos2x}}\right)\left(\frac{\sin^2x}{x^2}\right)=\frac{3}{2}$

10. Since f is continuous at $x = 0$

$\therefore$ $\lim_{x\to0^-}f(x)=f(0)=\lim_{x\to0^+}f(x)$

$\Rightarrow$ $e^p=q=e^{3/5}$ $\Rightarrow$ $p=\frac{3}{5}$, $q=e^{3/5}$

concept BOOSTERS

Class XI

Conic Sections

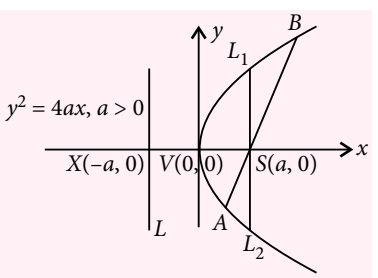
This column is aimed at Class XI students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

CONIC

- A conic is the locus of a point whose distance from a fixed point bears a constant ratio to its distance from a fixed line.
- The fixed point is called focus S. The fixed line is called directrix L.
- The constant ratio is the eccentricity e.
- (i) If $0 < e < 1$, the conic is an ellipse.
  (ii) If $e = 1$, the conic is a parabola.
  (iii) If $e > 1$, the conic is a hyperbola.
- If $S = (x_1, y_1)$, L is $ax + by + c = 0$ and e is the eccentricity, then the conic is

$$\sqrt{(x - x_1)^2 + (y - y_1)^2} = \frac{e\,|ax + by + c|}{\sqrt{a^2 + b^2}}.$$

PARABOLA



- Standard equation is $y^2 = 4ax$, $a > 0$.
- The curve is symmetric about the x-axis.
- It is the axis of the parabola.
- The curve opens to the right side.
- The axis cuts the directrix L at $X(-a, 0)$.
- $V(0, 0)$ is the vertex.
- $S(a, 0)$ is the focus.
- The tangent at the vertex V is $x = 0$.
- Focal chord is a chord AB through the focus S.
- Latus rectum is $L_1\,L_2$, the focal chord perpendicular to the x-axis.
- The ends of latus rectum are $L_1(a, 2a)$ and $L_2(a, -2a)$.
- The length of the latus rectum is $4a$.

NOTATION

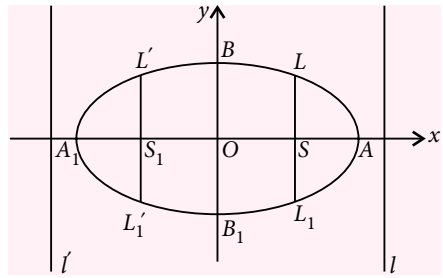
- $S = y^2 - 4ax$, $S_1 = y_1y - 2a(x + x_1)$,
  $S_{11} = y_1^2 - 4ax_1$, $S_2 = y_2y - 2a(x + x_2)$,
  $S_{12} = y_1y_2 - 2a(x_1 + x_2)$.
- Location of point $P(x_1, y_1)$:
  P is inside if $S_{11} < 0$
  P is outside if $S_{11} > 0$
  P is on the parabola if $S_{11} = 0$
- Tangent at $P(x_1, y_1) : S_1 = 0$
  The line $y = mx + c$ intersects the parabola at two points if $cm > a$.
  It does not intersect if $cm < a$.
  It is a tangent if $c = \frac{a}{m}$.
  The tangent with slope m is $y = mx + \frac{a}{m}$.
  It is tangent at $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$ on the parabola.
- Normal at $P(x_1, y_1) : y_1x + 2ay = x_1y_1 + 2ay_1$

- Chord joining two points (x_1, y_1) and (x_2, y_2) : $S_1 + S_2 = S_{12}$.
- Chord with midpoint (x_1, y_1) : $S_1 = S_{11}$
- Chord of contact of tangents from (x_1, y_1) : $S_1 = 0$
- Pair of tangents from (x_1, y_1) : $S_1^2 = S_{11}S$

PARAMETRIC FORM OF $y^2 = 4ax$

- $x = at^2, y = 2at$
- Tangent at the point, t : $x - yt + at^2 = 0$, slope $= \frac{1}{t}$.
- Tangents at t_1 and t_2 meet at the point $(at_1t_2, a(t_1 + t_2))$.
- Tangents at t_1 and t_2 are perpendicular $t_1t_2 = -1$.
- Tangents at t_1 and t_2 meet on the directrix $t_1t_2 = -1$.
- Normal at t : $xt + y = at^3 + 2at$, slope $= -t$.
- Normal at t_1 and t_2 meet at the point : $(a(2 + t_1^2 + t_1t_2 + t_2^2), -at_1t_2(t_1 + t_2))$
- Normal at t_1 meets the curve again at t_2 : $t_2 = -t_1 - \frac{2}{t_1}$.
- Normals at t_1 and t_2 intersect at t_3 on the curve : $t_1t_2 = 2, t_1 + t_2 + t_3 = 0, t_3^2 \geq 8$.
- Normals at t_1, t_2, t_3 are concurrent, $t_1 + t_2 + t_3 = 0$.
  The circle through the points t_1, t_2, t_3 passes through the origin.
- Let t_1, t_2, t_3 be the feet of normals drawn from (h, k) :
  $h > 2a, t_1 + t_2 + t_3 = 0, k = 0$: x-axis a normal
  $h = 3a, k = 0$: two of the normals are orthogonal.
- Chord joining the points t_1 and t_2 : $2x - (t_1 + t_2)y + 2at_1t_2 = 0$, slope $= \frac{2}{t_1 + t_2}$.
- Chord joining t_1 and t_2 is a focal chord : $t_1t_2 = -1$.
- Chord joining t_1 and t_2 subtends 90° at the origin : $t_1t_2 = -4$.

ELLIPSE



- Standard equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, 0 < a < b$
- The curve is symmetric about x-axis and y-axis.
- It is enclosed in the rectangle formed by the lines $x = \pm a$ and $y = \pm b$.
- Centre $= O(0, 0)$.
- Major axis $= 2a, A(a, 0), A_1(-a, 0)$.
- Minor axis $= 2b, B(0, b), B_1(0, -b)$.
- Foci $S\ (ae, 0), S_1\ (-ae, 0)$.
- Latus rectum $LL_1 = \frac{2b^2}{a}$,

$$L\left(ae, \frac{b^2}{a}\right), L_1\left(ae, \frac{-b^2}{a}\right)$$

- Latus rectum $L'L_1' = \frac{2b^2}{a}$,

$$L'\left(-ae, \frac{b^2}{a}\right), L_1'\left(-ae, \frac{-b^2}{a}\right)$$

- Directrices $l : x = \frac{a}{e}, l' : x = -\frac{a}{e}$

$$\Rightarrow\ e = \sqrt{1 - \frac{b^2}{a^2}}$$

- The major and minor axes are along x-axis and y-axis respectively.

NOTATION

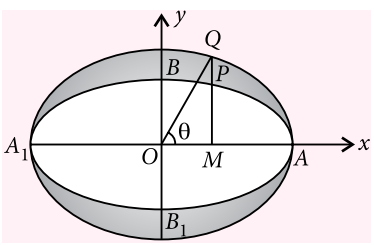
$$S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1, \qquad S_1 \equiv \frac{x_1x}{a^2} + \frac{y_1y}{b^2} - 1,$$

$$S_{11} \equiv \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1, \quad S_2 \equiv \frac{x_2x}{a^2} + \frac{y_2y}{b^2} - 1,$$

$$S_{22} \equiv \frac{x_2^2}{a^2} + \frac{y_2^2}{b^2} - 1, \quad S_{12} \equiv \frac{x_1x_2}{a^2} + \frac{y_1y_2}{b^2} - 1.$$

- Location of $P(x_1, y_1)$:
  P is inside the ellipse, if $S_{11} < 0$.
  P is outside the ellipse, if $S_{11} > 0$.
  P is on the ellipse, if $S_{11} = 0$.
- Chord with midpoint $P(x_1, y_1)$: $S_1 = S_{11}$
- Chord joining the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ on the ellipse : $S_1 + S_2 = S_{12}$
- Tangent at $P(x_1, y_1)$: $S_1 = 0$
  The line $y = mx + c$ intersects the ellipse at two points if $c^2 < a^2m^2 + b^2$, does not intersect if $c^2 > a^2m^2 + b^2$ and is a tangent if $c^2 = a^2m^2 + b^2$.
  The tangents with slope m are
  $y = mx \pm \sqrt{a^2m^2 + b^2}$.
- Chord of contact of tangents from $P(x_1, y_1)$: $S_1 = 0$
- Pair of tangents from $P(x_1, y_1)$: $S_1^2 = S_{11}S$
- Normal at $P(x_1, y_1)$: $\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$.

PARAMETRIC FORM OF $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



- $x = a\cos\theta$, $y = b\sin\theta$, θ is the eccentric angle.
- Tangent at P is $S_1 = 0$: $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$.
- Normal at P : $\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$

$$\therefore \quad \frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2 = a^2e^2$$

- Chord joining two points :
  $P = (a\cos\theta_1, b\sin\theta_1)$, $Q = (a\cos\theta_2, b\sin\theta_2)$
  The chord PQ is $S_1 + S_2 = S_{12}$

$$\Rightarrow \quad \frac{(x_1 + x_2)x}{a^2} + \frac{(y_1 + y_2)y}{b^2} = \frac{x_1x_2}{a^2} + \frac{y_1y_2}{b^2} + 1$$

$$\Rightarrow \quad (\cos\theta_1 + \cos\theta_2)\frac{x}{a} + (\sin\theta_1 + \sin\theta_2)\frac{y}{b}$$

$$= \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 + 1$$

$$= \cos(\theta_1 - \theta_2) + 1.$$

$$\text{or} \quad \frac{x}{a}\cos\left(\frac{\theta_1 + \theta_2}{2}\right) + \frac{y}{b}\sin\left(\frac{\theta_1 + \theta_2}{2}\right)$$

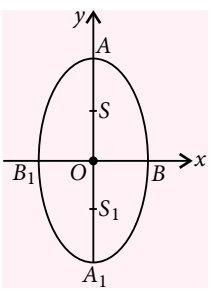
$$= \cos\left(\frac{\theta_1 - \theta_2}{2}\right).$$

OTHER ORIENTATIONS OF ELLIPSE

- $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $0 < a < b$. $AA_1 = 2b$ and $BB_1 = 2a$
  Foci : $\left(0, \pm\sqrt{b^2 - a^2}\right)$

$$e = \sqrt{1 - \left(\frac{a}{b}\right)^2}, b^2 - a^2 = b^2e^2$$

Directrices : $y = \pm\frac{b}{e}$



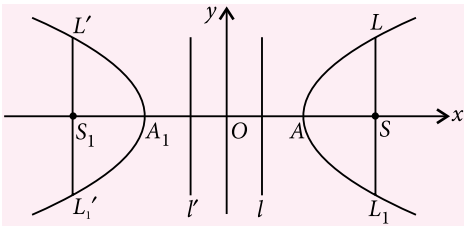
- $\frac{(x - \alpha)^2}{a^2} + \frac{(y - \beta)^2}{b^2} = 1$, $0 < b < a$
  Centre (α, β) and Foci $(\alpha \pm ae, \beta)$

$$e = \sqrt{1 - \frac{b^2}{a^2}}, \; a^2 - b^2 = a^2e^2$$

Directrices : $x = \alpha \pm \frac{a}{e}$

Major axis $= 2a$, Minor axis $= 2b$.

HYPERBOLA



- Standard equation is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
- The curve is symmetric about the x-axis and also about the y-axis. It has two branches and lies in the region $|x| \geq a$.
- The eccentricity,

  $e = \sqrt{1 + \left(\frac{b}{a}\right)^2}$, $b^2 = a^2\,(e^2 - 1)$.
- Centre : $O(0, 0)$
- Foci : $S(ae, 0)$, $S_1(-ae, 0)$.
  Vertices : $A(a, 0)$, $A_1(-a, 0)$

  Directrices : $l : x = \frac{a}{e}$, $l' : x = -\frac{a}{e}$
- Transverse axis is $AA_1 = 2a$,
  Conjugate axis is $2b$
- Latus rectum : $LL_1 = L'L_1{}' = \frac{2b^2}{a} = 2a(e^2 - 1)$
- Asymptotes : The lines $\frac{x^2}{a^2} = \frac{y^2}{b^2}$
- If the angle between them is 2α, then $e = \sec\alpha$.

NOTATION

$$S \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1, \qquad S_1 \equiv \frac{x_1 x}{a^2} - \frac{yy_1}{b^2} - 1$$

$$S_{11} \equiv \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1, \qquad S_2 \equiv \frac{x_2 x}{a^2} - \frac{y_2 y}{b^2} - 1$$

$$S_{22} = \frac{x_2^2}{a^2} - \frac{y_2^2}{b^2} - 1, \qquad S_{12} = \frac{x_1 x_2}{a^2} - \frac{y_1 y_2}{b^2} - 1.$$

- Location of $P(x_1, y_1)$: P is inside if $S_{11} > 0$
  P is outside if $S_{11} < 0$, P is on the curve if $S_{11} = 0$.
- Chord with midpoint $P(x_1, y_1)$: $S_1 = S_{11}$
- Chord joining $P(x_1, y_1)$ and $Q(x_2, y_2)$: $S_1 + S_2 = S_{12}$.
- Tangent at $P(x_1, y_1)$: $S_1 = 0$

  The line $y = mx + c$ intersects the hyperbola at two points if $c^2 > a^2m^2 - b^2$, does not intersect if $c^2 < a^2m^2 - b^2$ and is a tangent if $c^2 = a^2m^2 - b^2$.
  The tangents with slope m are
  $y = mx \pm \sqrt{a^2m^2 - b^2}$.
- Chord of contact of tangents from $P(x_1, y_1)$:

  $S_1 = 0$, $\frac{x_1 x}{a^2} - \frac{y_1 y}{b^2} = 1$.
- Pair of tangents from $P(x_1, y_1)$: $S_1{}^2 = S_{11}S$

$$\left(\frac{x_1 x}{a^2} - \frac{y_1 y}{b^2} - 1\right)^2 = \left(\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1\right)\left(\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1\right).$$

- Normal at $P(x_1, y_1)$:

  $\frac{a^2 x}{x_1} + \frac{b^2 y}{y_1} = a^2 + b^2$

PARAMETRIC FORMS

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

(i) $x = \frac{a}{2}\left(t + \frac{1}{t}\right)$, $y = \frac{b}{2}\left(t - \frac{1}{t}\right)$

(ii) $x = a\cosh t$, $y = b\sinh t$

(iii) $x = a\sec\theta$, $y = b\tan\theta$.

- Tangent : $\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$,
- Normal : $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2 = a^2e^2$
- Chord joining θ_1 and θ_2 :

  $\frac{x}{a}\cos\left(\frac{\theta_1 - \theta_2}{2}\right) - \frac{y}{b}\sin\left(\frac{\theta_1 + \theta_2}{2}\right)$

  $= \cos\left(\frac{\theta_1 + \theta_2}{2}\right)$

  It is a focal chord if $\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2} = \frac{1+e}{1-e}$

  or $\frac{1-e}{1+e}$

RECTANGULAR HYPERBOLA

If the curve $x^2 - y^2 = a^2$ is rotated through an angle $\frac{\pi}{4}$ about the origin and replacing a^2 by $2c^2$, we get the rectangular hyperbola $xy = c^2$ with centre $(0, 0)$, eccentricity $\sqrt{2}$, Vertices $(c, c), (-c, -c)$. Foci $\left(\sqrt{2}c, \sqrt{2}c\right), \left(-\sqrt{2}c, -\sqrt{2}c\right)$,

Latus rectum $= 2\sqrt{2}c$

Asymptotes $x = 0, y = 0$, Axes $x - y = 0, x + y = 0$

Directrices $x + y = \pm\sqrt{2}c$.

NOTATION

$S = xy - c^2$

$S_1 = \frac{x_1 y + x y_1}{2} - c^2$, $S_{11} = x_1 y_1 - c^2$,

$S_2 = \frac{x_2 y + x y_2}{2} - c^2$, $S_{12} = \frac{x_1 y_2 + x_2 y_1}{2} - c^2$.

- Tangents at $(x_1, y_1) : S_1 = 0 \Rightarrow x_1 y + x y_1 = 2c^2$
  Tangent with slope m : $y = mx \pm 2c\sqrt{-m}$
- Normal at $(x_1, y_1) : x_1 x - y_1 y = x_1^2 - y_1^2$.
- Chord with midpoint $(x_1, y_1) : S_1 = S_{11}$
- Chord of contact of tangents from (x_1, y_1) : $S_1 = 0$
- Pair of tangents from $(x_1, y_1) : S_1^2 = S_{11} S$

(vi) Chord joining the points $(x_1, y_1), (x_2, y_2)$: $S_1 + S_2 = S_{12}$

PARAMETRIC FORM OF $xy = c^2$

- $x = ct^2, \ y = \frac{c}{t}$
- Tangent at $t : S_1 = 0 \Rightarrow x + yt^2 = 2ct$
- Normal at $t : \ t^2 x - y = c\left(t^3 - \frac{1}{t}\right)$
- Chord joining t_1 and t_2

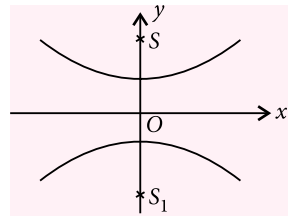
$$\left(\frac{1}{t_1} + \frac{1}{t_2}\right)x + (t_1 + t_2)y = \left(\frac{t_1}{t_2} + \frac{t_2}{t_1} + 2\right)c$$

- If a circle cuts $xy = c^2$ at four points $\left(ct_i, \frac{c}{t_i}\right)$, $i = 1, 2, 3, 4$, then
  $t_1 t_2 t_3 t_4 = 1$, $x_1 x_2 x_3 x_4 = y_1 y_2 y_3 y_4 = c^4$

- The orthocentre of a triangle inscribed in $xy = c^2$ lies on the curve.
  If t_1, t_2, t_3, t_4 represents the vertices of the triangle and the orthocentre, then $t_1 t_2 t_3 t_4 = -1$.

CONJUGATE HYPERBOLA

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$$



Eccentricity, $e_1 = \sqrt{1 + \left(\frac{a}{b}\right)^2}$

The eccentricity of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is e is related to e_1 as $\frac{1}{e^2} + \frac{1}{e_1^2} = 1$.

Both the hyperbolas have the same asymptotes $\frac{x^2}{a^2} = \frac{y^2}{b^2}$.

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PROBLEMS

SECTION-I

Single Correct Answer Type

1. The asymptote of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ with any tangent to the hyperbola form a triangle whose area is $a^2\tan\lambda$ in magnitude then its eccentricity is

(a) $\sec\lambda$ (b) $\operatorname{cosec}\lambda$ (c) $\sec^2\lambda$ (d) $\operatorname{cosec}^2\lambda$

2. If the eccentricity of the hyperbola $x^2 - y^2\sec^2\alpha = 5$ is $\sqrt{3}$ times the eccentricity of the ellipse $x^2\sec^2\alpha + y^2 = 25$, then value of α is

(a) $\pi/6$ (b) $\pi/4$ (c) $\pi/3$ (d) $\pi/2$

3. Locus of the middle points of the parallel chords with gradient m of the rectangular hyperbola $xy = c^2$ is

(a) $y + mx = 0$ (b) $y - mx = 0$
(c) $my - x = 0$ (d) $my + x = 0$

4. If PN is the perpendicular from a point on a rectangular hyperbola $x^2 - y^2 = a^2$ on any of its asymptotes, then the locus of the mid point of PN is

(a) a circle (b) a parabola
(c) an ellipse (d) a hyperbola

5. The equation to the chord joining two points (x_1, y_1) and (x_2, y_2) on the rectangular hyperbola $xy = c^2$ is

(a) $\frac{x}{x_1+x_2}+\frac{y}{y_1+y_2}=1$

(b) $\frac{x}{x_1-x_2}+\frac{y}{y_1-y_2}=1$

(c) $\frac{x}{y_1+y_2}+\frac{y}{x_1+x_2}=1$

(d) $\frac{x}{y_1-y_2}+\frac{y}{x_1-x_2}=1$

6. If $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$ & $S(x_4, y_4)$ are 4 concyclic points on the rectangular hyperbola $xy = c^2$, the co-ordinates of the orthocentre of the triangle PQR are

(a) $(x_4, -y_4)$ (b) (x_4, y_4)
(c) $(-x_4, -y_4)$ (d) $(-x_4, y_4)$

7. The chord PQ of the rectangular hyperbola $xy = a^2$ meets the axis of x at A; C is the mid point of PQ & 'O' is the origin. Then the ΔACO is:

(a) equilateral (b) isosceles
(c) right angled (d) right isosceles.

8. A conic passes through the point (2, 4) and is such that the segment of any of its tangents at any point contained between the co-ordinate axes is bisected at the point of tangency. Then the foci of the conic are

(a) $(2\sqrt{2}, 0)$ and $(-2\sqrt{2}, 0)$

(b) $(2\sqrt{2}, 2\sqrt{2})$ and $(-2\sqrt{2}, -2\sqrt{2})$

(c) (4, 4) and (–4, –4)

(d) $(4\sqrt{2}, 4\sqrt{2})$ and $(-4\sqrt{2}, -4\sqrt{2})$

9. Length of latus rectum of the conic satisfying the differential equation, $xdy + ydx = 0$ and passing through the point (2, 8) is

(a) $4\sqrt{2}$ (b) 8 (c) $8\sqrt{2}$ (d) 16

10. Two parabolas have the same focus namely at the point (3, –28). If their directrices are the x-axis & y-axis respectively and slope of common chord is 't' then $|t|$ =

(a) 1 (b) 2 (c) 3 (d) 4

11. Let $P \neq (0, 0)$ be a point on the parabola $y = x^2$. The normal to the parabola at p meets the parabola at another point Q. The coordinates of the point 'P' so that area bounded by the normal line and the parabola is minimum.

(a) $\left(\frac{1}{4}, \frac{1}{16}\right)$ (b) $\left(\frac{1}{2}, \frac{1}{4}\right)$

(c) $\left(\frac{1}{3}, \frac{1}{9}\right)$ (d) $\left(\frac{2}{3}, \frac{4}{9}\right)$

12. The focus of a parabola whose axis is parallel to x-axis is $S(3, 1)$. The normal at $P(h, k)$ is $x+2y-10=0$ and M is the foot of the perpendicular from P to the directrix then Area of ΔSPM is

(a) 10 square units (b) 20 square units
(c) 25 square units (d) 50 square units

13. Area of the triangle formed by the tangents at the points (4, 6), (10, 8) and (2, 4) on the parabola $y^2 - 2x = 8y - 20$, is (in square units)

(a) 4 (b) 2 (c) 1 (d) 8

14. Tangents are drawn to $y^2 = 4ax$ from a variable point P moving on $x + a = 0$, then the locus of foot of perpendicular drawn from P on the chord of contact of P is

(a) $y = 0$ (b) $(x - a)^2 + y^2 = a^2$
(c) $(x - a)^2 + y^2 = 0$ (d) $y(x - a) = 0$

15. A line from $(-1, 0)$ intersects the parabola $x^2 = 4y$ at A & B. Then the locus of centroid of ΔOAB is

(a) $3x^2 - 2x = 4y$ (b) $3y^2 - 2y = 4x$
(c) $3x^2 + 2x = 4y$ (d) none of these

16. The line $x - b + \lambda y = 0$ cuts the parabola $y^2 = 4ax (a > 0)$ at $P(t_1)$ & $Q(t_2)$. If $b \in [2a, 4a]$ then range of t_1t_2 where $\lambda \in R$, is

(a) $[-4, -2]$ (b) $[2, 4]$
(c) $[4, 16]$ (d) $[-16, -4]$

17. If BC is a latus rectum of parabola $y^2 = 4ax$ and A is the vertex. Then the minimum length of the projection of BC on a tangent drawn in the portion BAC is

(a) $\sqrt{2}a$ (b) $2\sqrt{2}a$ (c) $2a$ (d) $3\sqrt{2}a$

18. Two distinct chords of the parabola $y^2 = 4ax$ passing through $(a, 2a)$ are bisected by the line $x + y = 1$. The length of the latus rectum of the parabola can be

(a) 9 (b) 3 (c) 4 (d) 5

19. A line bisecting the ordinate PN of a point $P(at^2, 2at)$, $t > 0$ on the parabola $y^2 = 4ax$ is drawn parallel to the axis to meet the curve at Q. If NQ meets the tangent at the vertex at the point T, then the coordinates of T are

(a) $\left(0, \frac{4}{3}at\right)$ (b) $(0, 2at)$
(c) $\left(\frac{1}{4}at^2, at\right)$ (d) $(0, at)$

20. Tangents are drawn from the points on the line $x - y - 5 = 0$ to $x^2 + 4y^2 = 4$, then all the chords of contact pass through a fixed point, whose co-ordinates are

(a) $\left(\frac{4}{5}, -\frac{1}{5}\right)$ (b) $\left(\frac{4}{5}, \frac{1}{5}\right)$
(c) $\left(-\frac{4}{5}, \frac{1}{5}\right)$ (d) $\left(-\frac{4}{5}, -\frac{1}{5}\right)$

21. The eccentricity of the ellipse $(x - 3)^2 + (y - 4)^2 = \frac{y^2}{9}$ is

(a) $\frac{\sqrt{3}}{2}$ (b) $\frac{1}{3}$ (c) $\frac{1}{3\sqrt{2}}$ (d) $\frac{1}{\sqrt{3}}$

22. For an ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ with vertices A and A', tangent drawn at the point P in the first quadrant meets the y-axis at Q and the chord $A'P$ meets the y-axis at M. If 'O' is the origin then $OQ^2 - MQ^2$ equals to

(a) 9 (b) 13 (c) 4 (d) 5

23. An ellipse and a hyperbola have the same centre origin, the same foci and the minor-axis of the one is the same as the conjugate axis of the other. If e_1, e_2 be their eccentricities respectively, then $e_1^{-2} + e_2^{-2}$ equals

(a) 1 (b) 2 (c) 3 (d) 4

24. The line, $lx + my + n = 0$ will cut the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in points whose eccentric angles differ by $\pi/2$ if

(a) $a^2l^2 + b^2n^2 = 2m^2$ (b) $a^2m^2 + b^2l^2 = 2n^2$
(c) $a^2l^2 + b^2m^2 = 2n^2$ (d) $a^2n^2 + b^2m^2 = 2l^2$

25. Point 'O' is the centre of the ellipse with major axis AB & minor axis CD. Point F is one focus of the ellipse. If $OF = 6$ & the diameter of the inscribed circle of triangle OCF is 2, then the product $(AB)\,(CD)$ is equal to

(a) 65 (b) 52
(c) 78 (d) none of these

26. Which one of the following is the common tangent to the ellipses, $\frac{x^2}{a^2 + b^2} + \frac{y^2}{b^2} = 1$ and $\frac{x^2}{a^2} + \frac{y^2}{a^2 + b^2} = 1$?

(a) $ay = bx + \sqrt{a^4 - a^2b^2 + b^4}$
(b) $by = ax - \sqrt{a^4 + a^2b^2 + b^4}$
(c) $ay = bx - \sqrt{a^4 + a^2b^2 + b^4}$
(d) $by = ax + \sqrt{a^4 - a^2b^2 + b^4}$

27. An ellipse is drawn with major and minor axes of lengths 10 and 8 respectively. Using one focus as centre, a circle is drawn that is tangent to the ellipse, with no part of the circle being outside the ellipse. The radius of the circle is

(a) $\sqrt{3}$ (b) 2 (c) $2\sqrt{2}$ (d) $\sqrt{5}$

SECTION-II

Multiple Correct Answer Type

28. Equations of a common tangent to the two hyperbolas $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$ is

(a) $y=x+\sqrt{a^2-b^2}$ (b) $y=x-\sqrt{a^2-b^2}$

(c) $y=-x+\sqrt{a^2-b^2}$ (d) $y=-x-\sqrt{a^2-b^2}$

29. If in rectangular hyperbola normal at any point P meet the axes in G and g, and C be the centre of hyperbola, then

(a) $PG = Pg$ (b) $PG = PC$

(c) $Pg = PC$ (d) $PG = 2.PC$

30. If foci of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ concide with the foci of $\frac{x^2}{25}+\frac{y^2}{9}=1$ and eccentricity of the hyperbola is 2, then

(a) $a^2 + b^2 = 16$

(b) there is no director circle to the hyperbola

(c) centre of the director circle is (0, 0)

(d) length of latus rectum of the hyperbola = 12

31. If the line $ax + by + c = 0$ is normal to $xy = 1$, then

(a) $a > 0, b > 0$ (b) $a > 0, b < 0$

(c) $b > 0, a < 0$ (d) $a < 0, b < 0$

32. If the circle $x^2 + y^2 = 1$ cuts the rectangular hyperbola $xy = 1$ in four points (x_i, y_i), $i = 1, 2, 3, 4$, then

(a) $x_1x_2x_3x_4 = 1$ (b) $y_1y_2y_3y_4 = 1$

(c) $x_1 + x_2 + x_3 + x_4 = 0$

(d) $y_1 + y_2 + y_3 + y_4 = 0$

33. A circle drawn on any focal chord of the parabola $y^2 = 4ax$ as diameter cuts the parabola at four points t_1, t_2, t_3, t_4 where t_1, t_2 are the extremity of diameter then

(a) $t_1t_2t_3t_4 = 3$ (b) $t_3t_4 = 3$

(c) $t_1t_2t_3t_4 = -3$ (d) $t_3t_4 = -1$

34. The equation, $3x^2 + 4y^2 - 18x + 16y + 43 = c$

(a) cannot represent a real pair of straight lines for any value of c

(b) represents an ellipse, if $c > 0$

(c) represent empty set, if $c < 0$

(d) a point, if $c = 0$

35. If the parabola $y = (a - b)x^2 + (b - c)x + (c - a)$ touches the x-axis in the interval (0, 2), then the line $ax + by + c = 0$

(a) always passes through a fixed point

(b) represents the family of parallel lines

(c) data insufficient

(d) passes through the point (–2, 1)

36. Two parabolas with same axis, focus of each being exterior to the other and the latus rectum being $4a$ and $4b$ are given. The locus of the midpoints of the intercepts between the parabolas made on the lines parallel to the common axis is

(a) Straight line if $a = b$

(b) Parabola if $a \neq b$

(c) Parabola for all a, b

(d) Ellipse if $b > a$

37. The ends of a line segment are $P(1, 3)$ and $Q(1,1)$. R is a point on the line segment PQ such that $PR : QR = 1: \lambda$. If R is an interior point of the parabola $y^2 = 4x$ then

(a) $\lambda \in (0, 1)$ (b) $\lambda \in \left(-\frac{3}{5}, 1\right)$

(c) $\lambda \in \left(\frac{1}{2}, \frac{3}{5}\right)$ (d) $\lambda \in \left(-\frac{3}{5}, 0\right)$

38. An ellipse whose distance between foci S and S' is 4 units is inscribed in the ΔABC touching the sides AB, AC and BC at P, Q and R. If centre of ellipse is at origin and major axis along x-axis $SP + S'P = 6$ then

(a) Equation of the ellipse is $\frac{x^2}{9}+\frac{y^2}{5}=1$

(b) If $\angle BAC = 90°$ then locus of vertex A is $x^2 + y^2 = 14$

(c) If chord $\overline{PQ}$ subtends 90° angle at centre of ellipse then locus of A is $25x^2 + 81y^2 = 630$

(d) Eccentricity of the ellipse is $\frac{2}{3}$

39. The tangent at any point 'P' on a standard ellipse with foci as S and S' meets the tangents at the vertices A and A' in the points V and V', then

(a) $l(AV).l(A'V') = b^2$

(b) $l(AV).l(A'V') = a^2$

(c) $\angle V'SV = 90°$

(d) $V'S'VS$ is a cyclic quadrilateral

40. A point on the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ at a distance equal to the mean of the lengths of the semi - major axis and semi-minor axis from the centre is

(a) $\left(\frac{2\sqrt{91}}{7},\frac{3\sqrt{105}}{14}\right)$ (b) $\left(\frac{2\sqrt{91}}{7},\frac{-3\sqrt{105}}{14}\right)$

(c) $\left(\frac{-2\sqrt{91}}{7},\frac{-3\sqrt{105}}{14}\right)$ (d) $\left(\frac{-2\sqrt{91}}{7},\frac{3\sqrt{105}}{14}\right)$

41. If a variable tangent to the circle $x^2 + y^2 = 1$ intersects the ellipse $x^2 + 2y^2 = 4$ at points P and Q, then the locus of the point of intersection of tangents to the ellipse at P and Q is a conic whose

(a) eccentricity is $\frac{\sqrt{3}}{2}$ (b) eccentricity is $\frac{\sqrt{5}}{2}$

(c) latus-rectum is of length 2 units

(d) foci are $(\pm 2\sqrt{5}, 0)$

SECTION-III

Comprehension Type

Paragraph for Question No. 42 to 44

The normal at any point (x_1, y_1) of curve is line perpendicular to tangent at (x_1, y_1). In case of parabola $y^2 = 4ax$ the equation of normal is $y = mx - 2am - am^2$ (m is slope of normal). In case of rectangular hyperbola $xy = c^2$ the equation of normal at $\left(ct, \frac{c}{t}\right)$ is $xt^3 - yt - ct^4 + c = 0$ and the shortest distance between any two curve always exist along the common normal

42. If normal at (5, 3) of hyperbola $xy - y - 2x - 2 = 0$ intersects the curve again at $(\alpha, \beta - 29)$, then $\frac{\beta}{\alpha}$ is

(a) 10 (b) 20 (c) 30 (d) 40

43. If the shortest distance between $2y^2 - 2x + 1 = 0$ and $2x^2 - 2y + 1 = 0$ is d then the number of solution of $|\sin\alpha| = 2\sqrt{2}d$ $(\alpha \in [-\pi, 2\pi])$ is

(a) 3 (b) 4

(c) 5 (d) none of these

44. Number of normals drawn from $\left(\frac{7}{6}, 4\right)$ to $y^2 = 2x - 1$

(a) 0 (b) 1 (c) 2 (d) 3

Paragraph for Question No. 45 to 47

Point $A(8,8)$ lies on the parabola $y^2 = 8x$. From point A, a focal chord is drawn which cuts the given parabola at point B. Normals at point A and point B intersect at point C. From point C, a normal is drawn (other than the above two) which cuts the parabola at point D.

45. Equation of normal at point B is

(a) $x+2y=\frac{9}{2}$ (b) $x-2y=\frac{9}{2}$

(c) $2y+x=\frac{9}{2}$ (d) $2y-x=\frac{9}{2}$

46. x coordinate of point D is

(a) $\frac{3}{2}$ (b) $\frac{5}{2}$ (c) $\frac{7}{2}$ (d) $\frac{9}{2}$

47. Coordinate of point C is

(a) $\left(\frac{9}{2}, 3\right)$ (b) $\left(\frac{17}{2}, 3\right)$ (c) $\left(\frac{21}{2}, 3\right)$ (d) $\left(\frac{13}{2}, 3\right)$

Paragraph for Question No. 48 to 50

Let 't' be a positive number. Draw two tangents from the point $(t, -1)$ to the parabola $y = x^2$. If the area bounded by the tangent lines and the parabola is denoted by $S(t)$ then

48. $S(t) =$

(a) $\frac{2}{3}(1+t^2)^{3/2}$ (b) $\frac{1}{3}(1+t^2)^{3/2}$

(c) $\frac{3}{4}(1+t^2)^{3/2}$ (d) $\frac{2}{3}(1+t^2)^{3/2}$

49. $\frac{S(t)}{\sqrt{t}}$ is minimum then $t =$

(a) $\frac{1}{\sqrt{5}}$ (b) $\sqrt{5}$ (c) $\frac{2}{\sqrt{3}}$ (d) $\frac{1}{\sqrt{3}}$

50. Minimum value of $\frac{S(t)}{\sqrt{t}}$ is

(a) $4\cdot(5)^{-1/4}\sqrt{6}$ (b) $4\cdot(5^{-5/4})\sqrt{6}$

(c) $\frac{16}{9\sqrt{3}}$ (d) $4\cdot(5^{-3/4})\sqrt{6}$

Paragraph for Question No. 51 to 52

$y = f(x)$ is a parabola of the form $y = x^2 + ax + 1$. Its tangent at the point of intersection of y-axis and the parabola also touch the circle $x^2 + y^2 = r^2$. It is known that no point of the parabola is below x-axis. Then answer the following.

51. The radius of the circle where 'a' attains its maximum value

(a) $\frac{1}{\sqrt{10}}$ (b) $\frac{1}{\sqrt{5}}$ (c) 1 (d) $\sqrt{5}$

52. The slope of the tangent when the radius of the circle is maximum

(a) 0 (b) 1

(c) –1 (d) not defined

SECTION-IV

Matrix-Match Type

53. Match the following

Column-I		Column-II	
(A)	Number of points on the curve $y = \|\|1 - e^x\| - 2\|$ from which mutually perpendicular tangents can be drawn to the parabola $x^2 + 4y = 0$	(p)	2
(B)	Locus of vertex of parabola whose focus is (1, 2) and length of latus rectum is 12 units is $(x - 1)^2 + (y - 2)^2 = a^2$, then a =	(q)	4
(C)	A line drawn through the focus F and parallel to tangent at $P(1, 2\sqrt{2})$ on the parabola $y^2 = 8x$ cut the line $y = 2\sqrt{2}$ at Q, then PQ is equal to	(r)	3
(D)	A variable parabola touches the x and y-axes at (1, 0) and (0, 1), then radius of locus of focus of parabola	(s)	0

54. Match the following.

Column-I		Column-II	
(A)	If P is a point on the ellipse $\frac{x^2}{16}+\frac{y^2}{20}=1$ whose foci are S, S', then $SP + S'P$.	(p)	$\frac{4}{5}$
(B)	The eccentricity of the ellipse $2x^2 + 3y^2 - 4x - 12y + 13 = 0$	(q)	$4\sqrt{5}$
(C)	Tangent are drawn from the points on the line $x - y - 5 = 0$ to $x^2 + 4y^2 = 4$, then all the chords of contact pass through a fixed point whose abscissa is	(r)	8
(D)	The sum of distance of any point on the ellipse $3x^2 + 4y^2 = 12$ from its directix is	(s)	$\frac{1}{\sqrt{3}}$

SECTION-V

Integer Answer Type

55. If a variable line has its intercepts on the coordinate axes e, e', where $\frac{e}{2}, \frac{e'}{2}$ are the eccentricities of a hyperbola and its conjugate hyperbola, then the line always touches the circle $x^2 + y^2 = r^2$, where r =

56. The number of possible tangents which can be drawn to the curve $4x^2 - 9y^2 = 36$, which are perpendicular to the straight line $5x + 2y - 10 = 0$ is

57. The number of points having integral coordinates lying outside the hyperbola $\frac{x^2}{13}-\frac{y^2}{8}=1$ from where two perpendicular tangents could be drawn to the hyperbola is

58. The graphs of $x^2 + y^2 + 6x - 24y + 72 = 0$ and $x^2 - y^2 + 6x + 16y - 46 = 0$ intersect at 4 points. The sum of the distances of these four points from the point (–3, 12) is d, then the number of positive divisors of d is

59. PQ and RS are two perpendicular chords of the rectangular hyperbola $xy = c^2$. If C is the centre of the rectangular hyperbola, then the product of the slopes of CP, CQ, CR and CS is

60. Let PN be the ordinate of a point P on the hyperbola $\frac{x^2}{(97)^2} - \frac{y^2}{(79)^2} = 1$ and the tangent at P meets the transverse axis at T, O is the origin. Then $\left[\frac{ON \cdot OT}{2011}\right]$ is equal to (where [.] denotes G.I.F)

61. From a point 'T' a tangent is drawn to touch the parabola $y^2 = 16x$ at $P(16, 16)$. If 'S' be the focus of the parabola, and $\angle TPS = \cot^{-1} k$, then k is equal to

62. Normals are drawn from the point P with slopes $m_1 \cdot m_2 = \alpha$ in a part of the parabola itself, then the value of α, is.

63. If ends of latus rectum of parabola are (2, 6) and (6, 2) and the equation of the possible directrix is $x + y = \lambda_i$, where $i = 1, 2$. Then the value of A.M. of λ_1 and λ_2 is

64. The locus of point from which 3 normals drawn to the curve $y^2 + 2y - 4x + 5 = 0$ are such that two of them are perpendicular is $(y - \mu/2)^2 - (x - \lambda) = 0$, then the value of $\lambda - 2\mu$ is equal to

65. The tangent at the point $P(1,1)$ on the curve $y = x^2 + bx - b$ meets the positive x and y-axes at A and B respectively and O is the origin. If b_1, b_2, b_3 are the values of b for which the area of the triangle OAB is 2 square units with $b_1 < b_2 < b_3$, then $|b_1| + |b_2 + b_3| =$

66. A circle with its centre at the focus of the parabola $y^2 = 4ax$ and touching its directrix intersects the parabola at points A, B. Then length AB is equal to ka. Then $k =$

67. An ellipse is drawn with major and minor axes of lengths 10 and 8 respectively. Using one focus as centre, a circle is drawn that is tangent to the ellipse, with no part of the circle being outside the ellipse. Find the radius of the circle.

68. A common tangent to $9x^2 + 16y^2 = 144$, $y^2 = x - 4$ and $(x - 6)^2 + y^2 = 4$ is $x = k$, then k is equal to

69. The area of the quadrilateral formed by the tangents at the end point of latus rectum to the ellipse $\frac{x^2}{9} + \frac{y^2}{5} = 1$ is $3K$. Then K is equal to

70. If a tangent of slope 2 of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is normal to the circle $x^2 + y^2 + 4x + 1 = 0$, then the maximum value of ab is

71. If line through $P(a, 2)$ meets the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ at A and D and meets the axes at B and C, so that PA, PB, PC, PD are in G.P., then minimum value of $|a|$ is

SOLUTIONS

1. (a) : $A = ab = a^2 \tan\lambda \Rightarrow b/a = \tan\lambda$

Hence $e^2 = 1 + (b^2/a^2) \Rightarrow e^2 = 1 + \tan^2\lambda$

$\Rightarrow e = \sec\lambda$

2. (b) : $\frac{x^2}{5} - \frac{y^2}{5\cos^2\alpha} = 1$

$e_1^2 = 1 + \frac{b^2}{a^2} = 1 + \frac{5\cos^2\alpha}{5} = 1 + \cos^2\alpha;$

Now, eccentricity of the ellipse $\frac{x^2}{25\cos^2\alpha} + \frac{y^2}{25} = 1$

is $e_2^2 = 1 - \frac{25\cos^2\alpha}{25} = \sin^2\alpha;$

put $e_1 = \sqrt{3}\, e_2 \Rightarrow e_1^2 = 3e_2^2$

$\Rightarrow 1 + \cos^2\alpha = 3\sin^2\alpha$

$\Rightarrow 2 = 4\sin^2\alpha \Rightarrow \sin\alpha = \frac{1}{\sqrt{2}}$

3. (b) : Equation of chord with mid point (h, k) is $\frac{x}{h} + \frac{y}{k} = 2;\ m = -\frac{k}{h} \Rightarrow y - mx = 0$

4. (d) : $P : (a\sec\theta, a\tan\theta);$

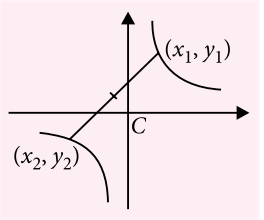
$N : [(a/2)(\sec\theta + \tan\theta),\ (a/2)(\sec\theta + \tan\theta)]$

$\Rightarrow 4h/a = 3\sec\theta + \tan\theta$ & $4k/a = \sec\theta + 3\tan\theta$

$\Rightarrow x^2 - y^2 = a^2/2$

5. (a) : Note that chord of $xy = c^2$ whose middle point is (h, k) is $\frac{x}{h} + \frac{y}{k} = 2$

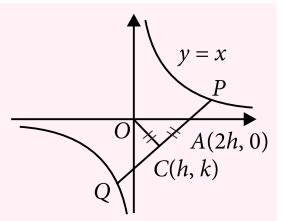
further, now $2h = x_1 + x_2$ and $2k = y_1 + y_2$



6. (c) : A rectangular hyperbola circumscribing a Δ also passes through its orthocentre if $\left(ct_i, \frac{c}{t_i}\right)$ (where i = 1, 2, 3) are the vertices of the Δ then therefore orthocentre is $\left(\frac{-c}{t_1t_2t_3}, -ct_1t_2t_3\right)$, where $t_1t_2t_3t_4 = 1$. Hence, orthocentre is $\left(-ct_4, \frac{-c}{t_4}\right) = (-x_4, -y_4)$

7. (b) : Chord with a given middle point $\frac{x}{h} + \frac{y}{k} = 2$

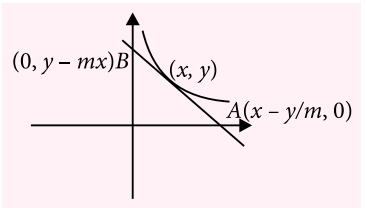
ΔACO is isosceles with $OC = CA$.]



8. (c) : T : $Y - y = m\,(X - x)$

$X = 0$, $Y = y - mx$

$Y = 0$, $X = x - \frac{y}{m}$ $\therefore$ $x - \frac{y}{m} = 2x$ $\Rightarrow$ $\frac{y}{m} = -x$



$\frac{dy}{dx} = -\frac{y}{x} \Rightarrow \frac{dy}{y} + \frac{dx}{x} = 0 \Rightarrow \ln xy = \ln c \Rightarrow xy = c$

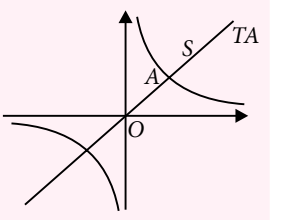
$x = 2$ & $y = 4$ $\Rightarrow$ equation is $xy = 8$

Solving with $y = x$

$x = 2\sqrt{2}$

$\therefore$ $OA = \sqrt{8+8} = 4$

$OS = 4\sqrt{2}$

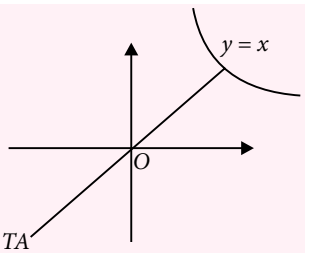


Coordinates of S are (4, 4) or (–4, – 4)

9. (c) : $\frac{dy}{y} + \frac{dx}{x} = 0$ $\Rightarrow$ $\ln xy = \ln c$ $\Rightarrow$ $xy = c$

passes through (2, 8) $\Rightarrow$ c = 16

xy = 16, Latus rectum = $2a(e^2 - 1) = 2a$



Solving with $y = x$, vertex is (4, 4)

Distance from centre to vertex = $4\sqrt{2}$

Length of latus rectum = length of TA = $8\sqrt{2}$

10. (a) : Equation of parabola whose directrix x-axis is $(x - 3)^2 = -56(y + 14)$ .... (1)

$\because$ $2a = 28$ $\Rightarrow$ $a = 14$ where vertex (3, –14)

Directrix parallel to y-axis $\Rightarrow$ $2a = 3 \Rightarrow a = 3/2$

$\Rightarrow$ vertex = $\left(\frac{3}{2}, -28\right)$

$\Rightarrow$ Equation of parabola

$$(y+28)^2 = 6\left(x - \frac{3}{2}\right) \quad \text{.... (2)}$$

Solving (1) and (2) we get $y = \pm x$

$\Rightarrow$ $|t| = 1$

11. (b) : Let (p, p^2) be a point on $y = x^2$ .... (1)

Equation of normal $y - p^2 = \frac{-1}{2p}(x - p)$ .... (2)

Solving (1) and (2) we get $q = -p - \frac{1}{2p}$

$$A = \int_q^p \left(p^2 - \frac{1}{2p}(x-p)\right) - x^2)dx$$

$\frac{dA}{dP} = 0 \Rightarrow p = \frac{1}{2}$ $\Rightarrow$ Point $P \equiv \left(\frac{1}{2}, \frac{1}{4}\right)$

12. (a) : Image of (3, 1) w.r.t. $x + 2y - 10 = 0$ is (5, 5)

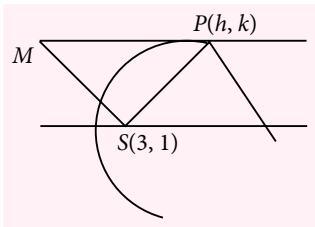
The horizontal line through (5, 5) is $y = 5$ meets the $x + 2y - 10 = 0$ at $P \Rightarrow P = (0, 5) \Rightarrow h + k = 5$

$SP = PM = 5$

$\Rightarrow$ Perpendicular distance from (0, 5) to $x + k = 0$ is 5 $\Rightarrow$ $k = 5$.

$\Rightarrow$ Equation of directrix is $x + 5 = 0$, $M = (-5, 5)$

$$\cos\angle SPM = \frac{25+25-80}{2\times5\times5} = \frac{-3}{5} \Rightarrow \sin\angle SPM = \frac{4}{5}$$



$$\text{Area of } \Delta SPM = \frac{1}{2}\times5\times5\times\frac{4}{5} = 10$$

13. (b) : Required area is half of the area of triangle formed by the points (4, 6), (10, 8), (2, 4)

$\therefore$ Required area = 2.

14. (c) : Portion of tangent intercepted between parabola and directrix subtends a right angle at the focus.

15. (c) : $A(2t_1, t_1^2)$, $B(2t_2, t_2^2)$

Again $P(-1, 0)$, A and B are collinear

$$\Rightarrow \quad \frac{t_2^2 - t_1^2}{2(t_2 - t_1)} = \frac{t_2^2}{2t_2 + 1}$$

$\Rightarrow (t_1 + t_2) = -2t_1t_2$ .... (i)

Let centroid is (h, k)

$$h = \frac{2(t_1 + t_2)}{3} \quad \text{.... (ii)}$$

$$k = \frac{t_1^2 + t_2^2}{3} \quad \text{.... (iii)}$$

From (i), (ii), & (iii) eliminate t_1 and t_2

$3h^2 + 2h = 4k$

Hence, locus is $3x^2 + 2x = 4y$.

16. (a) : Line PQ is $(t_1 + t_2)y = 2(x + at_1t_2)$ ...(1)

$\lambda y = -x + b$ ...(2)

(1) and (2) are same

$$\frac{t_1 + t_2}{\lambda} = \frac{2}{-1} = \frac{2at_1t_2}{b}$$

$\therefore \quad b = -at_1t_2$ and $2a \le -at_1t_2 \le 4a$

$\therefore \quad -4 \le t_1t_2 \le -2$

17. (b) : Let tangent at $P(at^2, 2at)$ make an angle θ with x-axis, then $\tan\theta = \frac{1}{t}$

Projection of BC on tangent

$$= BC\sin\theta = \frac{4a}{\sqrt{1+t^2}} \ge 2a\sqrt{2} \qquad (\text{as } -1 \le t \le 1)$$

18. (b) : Any point on $x + y = 1$ is $(t, 1 - t)$:

Equation of chord with this point $(t, 1 - t)$ is

$y(1 - t) - 2a(x + t) = (1 - t)^2 - 4at$

It passes through $(a, 2a)$

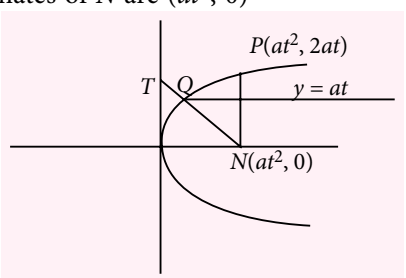
$\Rightarrow \quad t^2 - 2t + 2a^2 - 2a + 1 = 0$

This should have two distinct real roots

$\Rightarrow \quad a^2 - a < 0 \quad \Rightarrow \quad 0 < a < 1 \quad \Rightarrow \quad 0 < 4a < 4$

19. (a) : Equation of the line parallel to the axis and bisecting the ordinate PN of the point $P(at^2, 2at)$ is $y = at$ which meet the parabola $y^2 = 4ax$ at the point $Q\left(\frac{1}{4}at^2, at\right)$

Coordinates of N are $(at^2, 0)$



Equation of NQ is

$$y - 0 = \frac{0 - at}{at^2 - \frac{1}{4}at^2}(x - at^2)$$

which meets the tangent at the vertex $x = 0$ at the point $y = \frac{4}{3}at$

20. (a) : Let $A(x_1, x_1 - 5)$ be a point on $x - y - 5 = 0$ then chord of contact of $x^2 + 4y^2 = 4$ w.r.t. A is $xx_1 + 4y(x_1 - 5) = 4$

$\Rightarrow \quad (x + 4y)x_1 - (20y + 4) = 0$

It is passes through a fixed point

$\therefore \quad x + 4y = 0$

and $20y + 4 = 0$ ($\because$ It is of the form $P + \lambda Q = 0$)

$\Rightarrow \quad y = -1/5$ and $x = 4/5$

then co-ordinates of fixed point is $(4/5, -1/5)$.

21. (b) : $9(x-3)^2 + 9(y-4)^2 = y^2$

$9(x-3)^2 + 8y^2 - 72y + 144 = 0$

$9(x-3)^2 + 8(y^2 - 9y) + 144 = 0$

$$9(x-3)^2 + 8\left[\left(y-\frac{9}{2}\right)^2 - \frac{81}{4}\right] + 144 = 0$$

$$\Rightarrow \quad 9(x-3)^2 + 8\left(y-\frac{9}{2}\right)^2 = 162 - 144 = 18$$

$$\Rightarrow \quad \frac{9(x-3)^2}{18} + \frac{8\left(y-\frac{9}{2}\right)^2}{18} = 1$$

$$\Rightarrow \quad \frac{(x-3)^2}{2} + \frac{\left(y-\frac{9}{2}\right)^2}{9/4} = 1$$

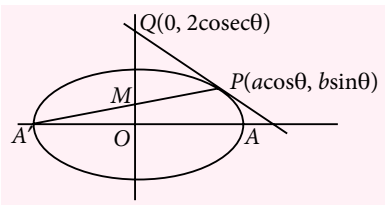
$$e^2 = 1 - \frac{2\cdot 4}{9} = \frac{1}{9}; \quad \therefore \ e = \frac{1}{3}$$

22. (c) : $a = 3$; $b = 2$

$$T : \frac{x\cos\theta}{3} + \frac{y\sin\theta}{2} = 1$$

$x = 0$; $y = 2\ \text{cosec}\theta$

Chord $A'P$, $y = \dfrac{2\sin\theta}{3(\cos\theta+1)}(x+3)$



Put $x = 0$, $y = \dfrac{2\sin\theta}{1+\cos\theta} = OM$

Now $OQ^2 - MQ^2 = OQ^2 - (OQ - OM)^2$

$= 2(OQ)(OM) - OM^2 = OM\{2(OQ) - (OM)\}$

$$= \frac{2\sin\theta}{1+\cos\theta}\left[\frac{4}{\sin\theta} - \frac{2\sin\theta}{1+\cos\theta}\right] = 4$$

23. (b) : $ae_1(E) = Ae_2(H)$ and $b^2 = a^2(1 - e_1^2)$

$= A^2(e_2^2 - 1)$.

Hence, $a^2 - a^2e_1^2 = A^2e_2^2 - A^2$.

Use the first relation.

24. (c) : Equation of a chord

$$\frac{x}{a}\cos\frac{\alpha+\beta}{2} + \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha-\beta}{2}$$

Put $\beta = \alpha + \dfrac{\pi}{2}$, equation reduces to,

$bx(\cos\alpha - \sin\alpha) + ay(\cos\alpha + \sin\alpha) = ab$ ... (1)

Compare with $lx + my = -n$ ... (2)

$$\left.\begin{aligned}\cos\alpha - \sin\alpha &= \frac{al}{-n}\\ \cos\alpha + \sin\alpha &= \frac{mb}{-n}\end{aligned}\right\}$$

Squaring and adding $a^2l^2 + b^2m^2 - 2n^2 = 0$

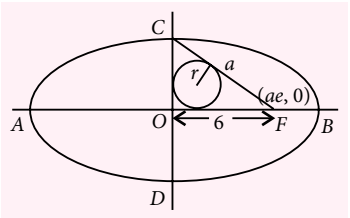
25. (a) : $a^2e^2 = 36 \Rightarrow a^2 - b^2 = 36$ ... (1)

Using $r = (s-a)\tan\dfrac{A}{2}$ in ΔOCF

$1 = (s - a)\tan 45°$ when $a = CF$

$2 = 2(s - a)$

$= 2s - 2a = 2s - AB$



$= (OF + FC + CO) - AB$

$$2 = 6 + \frac{AB}{2} + \frac{CD}{2} - AB$$

$$\frac{AB - CD}{2} = 4 \ \Rightarrow \ 2(a-b) = 8$$

$\Rightarrow \ a - b = 4$ ... (2)

From (1) and (2) $a + b = 9 \Rightarrow 2a = 13$; $2b = 5$

$\Rightarrow (AB)(CD) = 65$

26. (b) : Equation of a tangent to

$$\frac{x^2}{a^2+b^2} + \frac{y^2}{b^2} = 1$$

$$y = mx \pm \sqrt{(a^2+b^2)m^2 + b^2} \quad \text{....(1)}$$

If (1) is also a tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{a^2+b^2} = 1 \ \text{ then}$$

$(a^2 + b^2)m^2 + b^2 = a^2m^2 + a^2 + b^2$

(using $c^2 = a^2m^2 + b^2$)

$$b^2m^2 = a^2 \Rightarrow m^2 = \frac{a^2}{b^2} \Rightarrow m = \pm\frac{a}{b}$$

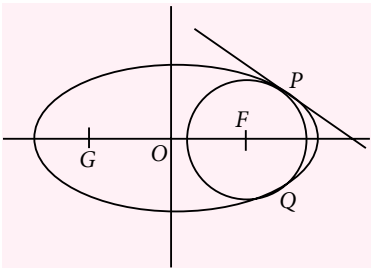
$$y = \pm\frac{a}{b}x \pm \sqrt{(a^2+b^2)\frac{a^2}{b^2}+b^2}$$

$$by = \pm ax \pm \sqrt{a^4+a^2b^2+b^4}$$

27. (b) : $2a = 10 \Rightarrow a = 5$; $2b = 8 \Rightarrow b = 4$

$$e^2 = 1-\frac{16}{25} = \frac{9}{25} \Rightarrow e = \frac{3}{5}$$

Focus = (3, 0)



Let the circle touches the ellipse at P and Q. Consider a tangent (to both circle and ellipse) at P. Let F(one focus) be the centre of the circle and other focus be G. A ray from F to P must retrace its path (normal to the circle). But the reflection property the ray FP must be reflected along PG. This is possible only if P, F and G are collinear. Thus P must be the end of the major axis.

Hence $r = a - ae = 5 - 3 = 2$

Alternately normal to an ellipse at P must pass through the centre (3, 0) of the circle

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$$

$$\Rightarrow \frac{5x}{\cos\theta} - \frac{4y}{\sin\theta} = 9 \quad \left(\theta \neq 0 \text{ or } \frac{\pi}{2}\right)$$

$$\frac{15}{\cos\theta} - 0 = 9 \Rightarrow \cos\theta = \frac{15}{9}$$

$\Rightarrow$ which is not possible $\Rightarrow \theta = 0$ or $\pi/2$

but $\theta \neq \pi/2 \Rightarrow \theta = 0$

Hence $P \equiv (5, 0)$ *i.e.* end of major axis

28. (a, b, c, d) : Given that, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ ...(1)

and $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ ...(2)

Tangent to Eq. (1), $y = mx \pm \sqrt{a^2m^2 - b^2}$

If this is also tangent to $\frac{x^2}{(-b^2)} - \frac{y^2}{(-a^2)} = 1$, then

$a^2m^2 - b^2 = (-b^2)m^2 - (-a^2) = a^2 - b^2m^2$

$(a^2 + b^2)m^2 = a^2 + b^2$

$m = \pm 1$

Hence, 4 common tangents are

$$y = \pm x \pm \sqrt{a^2 - b^2}.$$

29. (a, b, c) : Let the equation of the rectangular hyperbola be $x^2 - y^2 = a^2$. If P be any point on it as $(a\sec\phi, a\tan\phi)$, then the equation to the normal at P will be given by

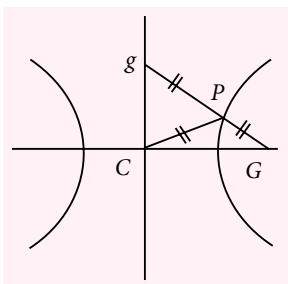
$x\sin\phi + y = 2a\tan\phi$

Its intersection with $y = 0$ gives $x = 2a\sec\phi$. As the point of intersection is G and C, the centre of the hyperbola. Hence, $CG = 2a\sec\phi$

Again the intersection of the normal with $x = 0$ gives $y = 2a\tan\phi$

If g denotes this point of intersection, we have

$Cg = 2a\tan\phi$



Now,

$PG^2 = (a\sec\phi - 2a\sec\phi)^2 + (a\tan\phi - 0)^2$

$= a^2(\sec^2\phi + \tan^2\phi)$

$Pg^2 = (a\sec\phi - 0)^2 + (a\tan\phi - 2a\tan\phi)^2$

$= a^2(\sec^2\phi + \tan^2\phi)$

and $PC^2 = (a\sec\phi - 0)^2 + (a\tan\phi - 0)^2$

$= a^2(\sec^2\phi + \tan^2\phi)$

Hence, $PG = Pg = PC$

30. (a, b, d) : For the ellipse,

$a = 5$ and $e = \sqrt{\frac{25-9}{25}} = \frac{4}{5}$

$\therefore \quad ae = 4$

Hence, the foci are $(-4, 0)$ and $(4, 0)$

For the hyperbola, $ae = 4$, $e = 2$

$\therefore \quad a = 2$

$b^2 = 4(4 - 1) = 12 \quad \Rightarrow \quad b = \sqrt{12}$

31. (b, c) : Slope of the normal at (x_1, y_1) of the rectangular hyperbola $xy = 1$ is $x_1^2 > 0$. So slope of $ax + by + c = 0$ which is $-a/b > 0 \Rightarrow a/b < 0$.

32. (a, b, c, d) : Let $(x_i, y_i) = (t_i, 1/t_i)$; $i = 1, 2, 3, 4$

Any point on the rectangular hyperbola $xy = 1$ is $(t, 1/t)$ which lies on the circle $x^2 + y^2 = 1$ if

$$t^2 + \frac{1}{t^2} = 1 \Rightarrow t^4 - t^2 + 1 = 0$$

The roots of the equation are t_1, t_2, t_3, t_4 where

$t_1 + t_2 + t_3 + t_4 = 0 \quad \Rightarrow \quad x_1 + x_2 + x_3 + x_4 = 0$

$\sum t_1 t_2 = -1, \ \sum t_1 t_2 t_3 = 0$

$t_1 t_2 t_3 t_4 = 1 \quad \Rightarrow \quad x_1 x_2 x_3 x_4 = y_1 y_2 y_3 y_4 = 1$

and $y_1 + y_2 + y_3 + y_4 = \frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \frac{1}{t_4}$

$$= \frac{\sum t_1 t_2 t_3}{t_1 t_2 t_3 t_4} = 0$$

33. (b, c)

34. (a, b, c, d) : $(\sqrt{3}x - 3\sqrt{3})^2 + (2y + 4)^2 = c$

Comparing this with

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

we get $h^2 < ab$

so no locus for $c < 0$

Ellipse for $c > 0$ and point for $c = 0$

35. (a, d) : Solving equation of parabola with x-axis (*i.e.* $y = 0$), we get

$(a - b)x^2 + (b - c)x + (c - a) = 0$ which should have two equal values of x, as x-axis touches the parabola

$\Rightarrow \quad (b - c)^2 - 4(a - b)(c - a) = 0$

$\Rightarrow \quad (b + c - 2a)^2 = 0 \quad \Rightarrow \quad b + c = 2a$

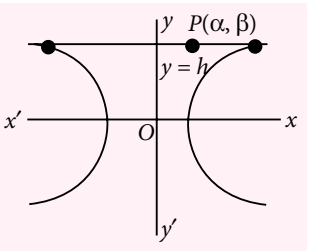
i.e. $ax + by + c = 0$ always pass through $(-2, 1)$

36. (a, b) : Given

$y^2 = 4a(x - k), \ y^2 = -4b(x + k)$

$A = \left(\frac{h^2}{4a} + k, h\right), \ B\left(-k - \frac{h^2}{4b}, h\right)$

Let $P(\alpha, \beta)$, then $\alpha = \frac{1}{2}\left(\frac{h^2}{4a} + k - k - \frac{h^2}{4b}\right), \beta = h$



$\therefore \quad 2\alpha = \frac{h^2}{4}\left(\frac{1}{a} - \frac{1}{b}\right).$

Locus of P is $2x = \frac{y^2}{2}\frac{(b-a)}{ab}$

if $a = b \quad \Rightarrow \quad x = 0$

if $a \neq b \Rightarrow \ y^2 = \frac{hab}{(b-a)} \ x$ which is parabola.

37. (a, b, c, d) :

$$R = \left(\frac{\lambda+1}{\lambda+1}, \frac{3\lambda+1}{\lambda+1}\right) = \left(1, 1 + \frac{2\lambda}{\lambda+1}\right)$$

$\Rightarrow \quad y_1^2 - 4x_1 < 0$

$$\Rightarrow \left(1 + \frac{2\lambda}{\lambda+1}\right)^2 - 4 < 0$$

$$\Rightarrow \left(1 + \frac{2\lambda}{\lambda+1} + 2\right)\left(1 + \frac{2\lambda}{\lambda+1} - 2\right) < 0$$

$$\Rightarrow \left(\frac{2\lambda}{\lambda+1} + 3\right)\left(\frac{2\lambda}{\lambda+1} - 1\right) < 0$$

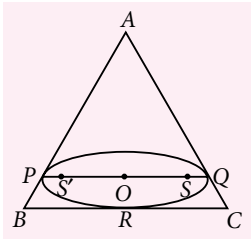
$$\Rightarrow \quad \frac{2\lambda}{\lambda+1} > -3 \text{ and } \frac{2\lambda}{\lambda+1} < 1$$

$\Rightarrow \quad 5\lambda > -3$ and $\lambda < 1$

$$\Rightarrow \quad \lambda > -\frac{3}{5} \text{ and } \lambda < 1 \quad \Rightarrow \quad \lambda \in \left(-\frac{3}{5}, 1\right)$$

38. (a, b, c, d) : (a) $2a = 6$, $2ae = 4$, $e = 2/3$,

$$E = \frac{x^2}{9} + \frac{y^2}{5} = 1$$



(b) A lies on the director circle of ellipse then locus of A is $x^2 + y^2 = 14$

(c) $A(h, k)$, equation of $PQ = \frac{xh}{9} + \frac{yk}{5} = 1$

Homogenesing the combined equation of OP and OQ is $\frac{x^2}{9} + \frac{y^2}{5} = \left(\frac{xh}{9} + \frac{yk}{5}\right)^2$

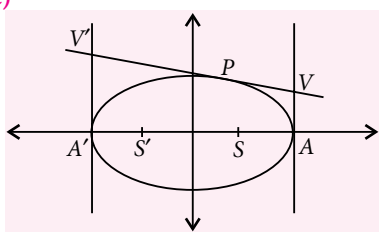
Now, $\angle POQ = 90°$ then
coefficient of x^2 + coefficient of $y^2 = 0$

$$\left(\frac{1}{9} - \frac{h^2}{9^2}\right) + \left(\frac{1}{5} - \frac{k^2}{5^2}\right) = 0$$

$$\Rightarrow 25x^2 + 81y^2 = 630 \text{ (locus of } (h, k))$$

(d) $e = 2/3$

39. (a, c)



Any tangent at $P(a\cos\theta, b\sin\theta)$ is

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$$

Line AV is $x = a$ and $A'V'$ is $x = -a$

So points $V \equiv \left(a, \frac{b(1-\cos\theta)}{\sin\theta}\right)$ and

$$V' \equiv \left(-a, \frac{b(1+\cos\theta)}{\sin\theta}\right)$$

So,

$$l(AV).\, l(A'V') = \frac{b(1-\cos\theta)}{\sin\theta} \times \frac{b(1+\cos\theta)}{\sin\theta} = b^2$$

Slope of line $V'S$ is $= \frac{b(1+\cos\theta)}{\sin\theta(-a-ae)} = m_1$

Also, slope of VS is $= \frac{b(1-\cos\theta)}{\sin\theta(a-ae)} = m_2$

Now, $m_1 \times m_2 = \frac{b^2(1-\cos^2\theta)}{\sin^2\theta(-b^2)} = -1$

$\Rightarrow \angle V'SV = 90°$ and $V'S'VS$ is not a cyclic quadrilateral at all.

40. (a, b, c, d) : Let $P(4\cos\theta, 3\sin\theta)$

$$\Rightarrow 16\cos^2\theta + 9\sin^2\theta = \frac{49}{4}$$

$$\Rightarrow \cos\theta = \pm\frac{\sqrt{91}}{14}, \sin\theta = \pm\frac{\sqrt{105}}{14}$$

41. (a, c) : A tangent to the circle $x^2 + y^2 = 1$ is $x\cos\theta + y\sin\theta = 1$. $R(x_0, y_0)$ is the point of intersection of the tangents to the ellipse at P and Q
$\Leftrightarrow x\cos\theta + y\sin\theta = 1$ and $x_0x + 2y_0y = 4$ represent the same line.
$\Leftrightarrow x_0 = 4\cos\theta$ and $y_0 = 2\sin\theta$

$$\Leftrightarrow \frac{x_0^2}{16} + \frac{y_0^2}{4} = 1$$

Hence, locus of R is the ellipse $\frac{x^2}{16} + \frac{y^2}{4} = 1$

42. (b) : Equation of hyperbola $(x - 1)(y - 2) = 4$
$\Rightarrow XY = 4$

If normal at $\left(ct, \frac{c}{t}\right)$ intersect curve again at $\left(ct', \frac{c}{t'}\right)$ then $t' = -\frac{1}{t^3}$

$2t = 4 \Rightarrow t = 2$

$$(X', Y') \equiv \left(-\frac{1}{4}, -16\right)$$

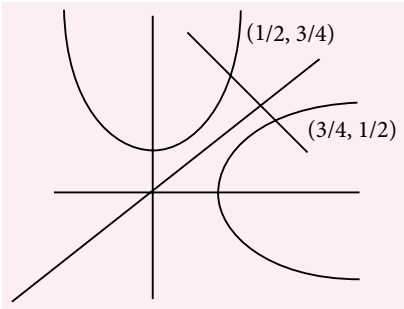
$$\Rightarrow (\alpha, \beta - 29) \equiv \left(\frac{3}{4}, -14\right) \Rightarrow \alpha = \frac{3}{4}, \beta = 15$$

so $\frac{\beta}{\alpha} = \frac{15 \times 4}{3} = 20$.

43. (a) : $2y\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{2y} = 1 \Rightarrow y = \frac{1}{2}$

So, $d = \sqrt{\frac{1}{16} + \frac{1}{16}} = \frac{1}{2\sqrt{2}}$

So $|\sin\alpha| = 1$
So number of solutions is 3.



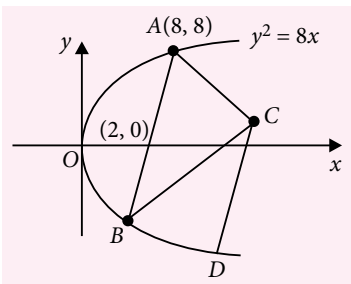
44. (b) : $y^2 = 2\left(x - \frac{1}{2}\right) \Rightarrow Y^2 = 2X$

For three normals $x - \frac{1}{2} > 1$

$x > \frac{3}{2}$

So only one normal can be drawn.

45. (b) **46. (d)** **47. (c)**



Let point be $A(t)$

$at^2 = 8$, $2at = 8$ and $a = 2$ so, $t = 2$

So, point $B(-1/2)$

i.e., $\left(2\left(\frac{-1}{2}\right)^2, 2\times 2\times\left(\frac{-1}{2}\right)\right) \equiv \left(\frac{1}{2}, -2\right)$

Slope of normal at 't' is $-t$

$\Rightarrow$ slope of normal at B is 1/2

So, equation of normal at B is $y + 2 = \frac{1}{2}\left(x - \frac{1}{2}\right)$;

i.e., $x - 2y = \frac{9}{2}$ ... (1)

Now, $y_A + y_B + y_D = 0$

$\Rightarrow 8 - 2 + y_D = 0 \Rightarrow y_D = -6$

So, $x_D = \frac{36}{8} = \frac{9}{2}$; So, $D \equiv \left(\frac{9}{2}, -6\right)$

Equation of normal at point A is

$y - 8 = -2(x - 8)$

i.e., $2x + y = 24$ .... (2)

Solving (1) and (2), C is $\left(\frac{21}{2}, 3\right)$.

48. (a) : Let (x_0, x_0^2) be any point on the curve

Slope of tangent $= m = \frac{x_0^2 + 1}{x_0 - t}$

Slope of tangent at 'x_0' $= 2x_0 = m$

$\Rightarrow x_0 = t \pm \sqrt{1 + t^2}$

Let $a = t - \sqrt{1 + t^2}$, $b = t + \sqrt{1 + t^2}$

Area enclosed by tangent lines and parabola is equivalent to the area below the parabola from $[a, b]$ and above the line $y = -1$ minus the area of two triangles first with vertices $(t, -1)$, (a, a^2), $(a, -1)$ and the second with vertices $(t, -1)$, (b, b^2), $(b, -1)$.

Sum of area of two triangles

$$= \frac{(a^2 + 1)(t - a)}{2} + \frac{(b^2 + 1)(b - t)}{2} = 2(1 + t^2)^{3/2}$$

$$S(t) = \int_a^b (x^2 + 1)\,dx - 2(1 + t^2)^{3/2}$$

$$= \frac{b^3 - a^3}{3} + (b - a) - 2(1 + t^2)^{3/2}$$

$$= \frac{8}{3}(1 + t^2)^{3/2} - 2(1 + t^2)^{3/2} = \frac{2}{3}(1 + t^2)^{3/2}$$

49. (a) : $\frac{S(t)}{\sqrt{t}}$ is minimum

$\Rightarrow \frac{d}{dt}\left(\frac{S(t)}{\sqrt{t}}\right) = 0 \Rightarrow t = \frac{1}{\sqrt{5}}$

50. (b) : $\text{Min}\left(\frac{S(t)}{\sqrt{t}}\right) = 4 \cdot 5^{-5/4} \cdot \sqrt{6}$

51. (b) **52. (a)**

Point of intersection is $P(0, 1)$

Tangent at P is $y = ax + 1$

Touches circle $\Rightarrow r^2 = \frac{1}{1 + a^2}$

Point does not lie below x-axis.

$\Rightarrow y \geq 0 \Rightarrow x^2 + ax + 1 \geq 0 \ \forall x$

$\Rightarrow a^2 - 4 \leq 0 \Rightarrow a \in [-2, 2]$

When a is maximum, $r = \frac{1}{\sqrt{1 + 4}} = \frac{1}{\sqrt{5}}$

Radius is maximum when $a = 0$

53. A - p ; B - r ; C - r ; D - s

(A) $y = ||1 - e^x| - 2|$ to $x^2 + 4y = 0$

Mutually perpendicular tangents to $x^2 + 4y = 0$ always intersects on $y = 1$.

$\therefore$ $||1 - e^x| - 2| = 1$ or $|1 - e^x| = 2 \pm 1$

$e^x = 2$ or $e^x = 4$

$\therefore$ two points exists.

(B) Distance between vertex and focus is $\frac{1}{4}(12) = 3$

(C) $y^2 = 8x$; $\left(\frac{dy}{dx}\right)_{(1,2\sqrt{2})} = \sqrt{2}$

$\therefore$ Focal chord is $y - 0 = \sqrt{2}(x - 2)$ meets $y = 2\sqrt{2}$

$\Rightarrow$ $Q = (4, 2\sqrt{2})$

$\therefore$ $PQ = 3$

(D) Focus is $\left(\frac{1}{2}, \frac{1}{2}\right)$ and is a fixed point.

$\therefore$ A circle of radius '0'.

54. A – q; B – s; C – p; D – r

(A) $PS + PS' = 2b = 4\sqrt{5}$

(B) gives ellipse

$$\frac{(x-1)^2}{\frac{1}{2}} + \frac{(y-2)^2}{\frac{1}{3}} = 1,\ a^2 = \frac{1}{2},\ b^2 = \frac{1}{3},\ e = \frac{1}{\sqrt{3}}$$

(C) Let $(t, t - 5)$ be any point on $x - y - 5 = 0$ the chord of contact to ellipse $x^2 + 4y^2 = 4$ is

$tx + 4(t - 5)y - 4 = 0$

$\Rightarrow$ $(-20y - 4) + t(x + 4y) = 0$

which pass through intersection of $x + 4y = 0$, $20y + 4 = 0$ *i.e.* $\left(\frac{4}{5}, \frac{-1}{5}\right)$

(D) $\frac{x^2}{4} + \frac{y^2}{3} = 1,\ a = 2,\ b = \sqrt{3},\ e = \frac{1}{2} \Rightarrow \frac{2a}{e} = 8$

55. (2) : Since $\frac{e}{2}$ and $\frac{e'}{2}$ are eccentricities of a hyperbola and its conjugate hyperbola, therefore

$$\frac{4}{e^2} + \frac{4}{e'^2} = 1 \Rightarrow 4 = \frac{e^2 e'^2}{e'^2 + e'^2}$$

The line passing through the points $(e, 0)$ and $(0, e')$ is

$e'x + ey - ee' = 0$

It is tangent to the circle $x^2 + y^2 = r^2$

Hence, $\frac{ee'}{\sqrt{e^2 + e'^2}} = r \Rightarrow r^2 = \frac{e^2 e'^2}{e^2 + e'^2} = 4$

$\Rightarrow$ $r = 2$

56. (0) : Tangent to $\frac{x^2}{9} - \frac{y^2}{4} = 1$ at $P(3\sec\theta, 2\tan\theta)$ is

$$\frac{x}{3}\sec\theta - \frac{y}{2}\tan\theta = 1$$

This is perpendicular to

$5x + 2y - 10 = 0$

$$\Rightarrow \frac{2\sec\theta}{3\tan\theta} = \frac{2}{5} \Rightarrow \sin\theta = \frac{5}{3}$$

which is not possible.

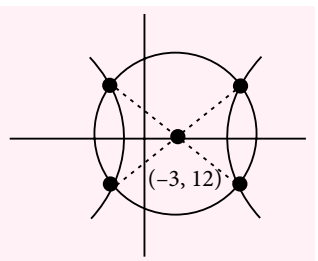
Hence, there is no such tangent.

57. (2) : Find the points on the circle $x^2 + y^2 = 5$.

i.e., $(\pm 1, \pm 2)$, $(\pm 2, \pm 1)$

58. (9) : Circle, hyperbola intersect at 4 points.

Given point = (–3, 12) = centre of circle



Radius = $\sqrt{9 + 144 - 72} = 9$

$\therefore$ Required sum = 9 + 9 + 9 + 9 = 36 = $2^2 \cdot 3^2$

No. of divisors = (2 + 1)(2 + 1) = 9

59. (1) : Let coordinates of P, Q, R, S be $\left(ct_i, \frac{c}{t_i}\right)$ respectively, i = 1, 2, 3, 4

Now $PQ \perp RS$

$$\Rightarrow \left(\frac{\frac{c}{t_2} - \frac{c}{t_1}}{ct_2 - ct_1}\right)\left(\frac{\frac{c}{t_4} - \frac{c}{t_3}}{ct_4 - ct_3}\right) = -1 \Rightarrow t_1 t_2 t_3 t_4 = -1$$

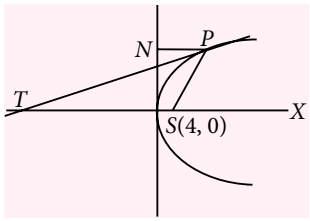
$\therefore$ Product of slopes = $\frac{1}{t_1^2}\frac{1}{t_2^2}\frac{1}{t_3^2}\frac{1}{t_4^2} = 1$

60. (4) : $ON \cdot OT = 97\cos\theta \cdot 97\sec\theta = 97^2$

$$\therefore \left[\frac{ON \cdot OT}{2011}\right] = \left[\frac{97^2}{2011}\right] = 4$$

61. (2) : Since PT bisects $\angle NPS$

$$\therefore \quad \angle NPS = \angle PSX = \tan^{-1}\frac{16}{12} = \tan^{-1}\frac{4}{3} = \theta.$$



$$\tan\theta = \frac{4}{3} = \frac{2\tan(\theta/2)}{1-\tan^2(\theta/2)} \Rightarrow \tan(\theta/2) = \frac{1}{2}$$

62. (2) : Let the point P be (h, k)

$\therefore$ Equation of the normal is

$k = mh - 2m - m^3$

$\Rightarrow \quad m^3 + m(2 - h) + k = 0$

$$\Rightarrow \quad m_1m_2m_3 = -k \Rightarrow m_3 = -\frac{k}{\alpha}$$

$$\Rightarrow \quad \left(-\frac{k}{\alpha}\right)^3 - \frac{k}{\alpha}(2-h) + k = 0$$

$\Rightarrow \quad k^2 = \alpha^2 h - 2\alpha^2 + \alpha^3$

Thus the locus of (h, k) is

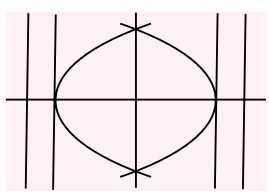
$y^2 = \alpha^2 x - 2\alpha^2 + \alpha^3$

On compaing it with $y^2 = 4x$, we get $\alpha^2 = 4$ and $-2\alpha^2 + \alpha^3 = 0 \quad \Rightarrow \quad \alpha = 2$

63. (8) : Length of latus rectum is $4\sqrt{2} = 4a$

$$\Rightarrow \quad a = \sqrt{2}$$

Equation of directrix is $x + y = \lambda$ whose distance from (4, 4) is $2\sqrt{2}$



$$\therefore \left|\frac{4+4-\lambda}{\sqrt{2}}\right| = 2\sqrt{2} \Rightarrow 8 - \lambda = \pm 4$$

$$\Rightarrow \quad \lambda = 4 \text{ or } 12 \quad \therefore \quad \frac{\lambda_1 + \lambda_2}{2} = 8$$

64. (8) : The curve is $(y + 1)^2 = 4(x - 1)$

Equation of the normal to the given curve is

$y + 1 = m(x - 1) - 2m - m^3$

which passes through (h, k)

$m^3 + m(3 - h) + 1 + k = 0$

$m_1m_2m_3 = -(1 + k) \quad \Rightarrow \quad m_3 = (1 + k)$

$(\because \ m_1m_2 = -1)$

$(1 + k)^3 + (1 + k)(3 - h) + (1 + k) = 0$

$(1 + k)^2 + 3 - h + 1 = 0$

$(y + 1)^2 = (x - 4)$

$$\therefore \quad \frac{\mu}{2} = -1 \Rightarrow \mu = -2, \lambda = 4$$

$\therefore \quad \lambda - 2\mu = 8$

65. (5) : $$m = \left(\frac{dy}{dx}\right)_{(1,\,1)} = b + 2$$

Equation of the tangent is

$(y - 1) = (b + 2)(x - 1)$

$\Rightarrow \quad (b + 2)x - y - (b + 1) = 0$

Now, area of the triangle $OAB = 2$

$$\Rightarrow \quad \frac{(b+1)^2}{2|b+2|} = 2 \quad \Rightarrow \quad b^2 + 2b + 1 = \pm 4(b + 2)$$

$\Rightarrow \quad b^2 - 2b - 7 = 0 \quad$ or $\quad b^2 + 6b + 9 = 0$

$$\Rightarrow \quad b_1 = -3,\ b_2 = 1 - 2\sqrt{2},\ b_3 = 1 + 2\sqrt{2}$$

$\Rightarrow \quad |b_1| + |b_2 + b_3| = 5$

66. (4) : Centre of circle $(a, 0)$ and radius $2a$

Equation of circle $(x - a)^2 + y^2 = 4a^2$

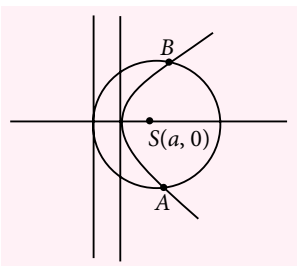
$x^2 + y^2 - 2ax - 3a^2 = 0$ and $y^2 = 4ax$

Solving, $x^2 + 4ax - 2ax - 3a^2 = 0$

$x^2 + 2ax - 3a^2 = 0$

$x = -3a, a$ and $y = \pm 2a$

$\therefore$ Length of $AB = 4a$



67. (2) : Focal distance of any point on ellipse $= a - ae\cos\theta$

$$e = \sqrt{1 - \frac{16}{25}} = \frac{3}{5}$$

So, circle is tangent at that point which is nearest to focus.

So radius of circle = minimum focal distance of point = $(a-ae)=5-5\times\frac{3}{5}=2$ units

68. (4) : Given $\frac{x^2}{16}+\frac{y^2}{9}=1$,

$y^2=x-4, (x-6)^2+y^2=4$

Hence $x = 4$ is indeed the common tangent of the given curve.

69. (9) : $e = 2/3$

Equation of tangent at L is $\frac{2x}{9}+\frac{y}{3}=1$. It meets x-axis at $A\left(\frac{9}{2},0\right)$ and y-axis at $B(0, 3)$.

$\therefore$ Required area $= 4\left[\frac{1}{2}\cdot\frac{9}{2}\cdot 3\right]=27$

70. (4) : A tangent of slope 2 is

$$y=2x\pm\sqrt{4a^2+b^2} \quad \ldots (1)$$

This is normal to the circle $x^2 + y^2 + 4x + 1 = 0$ *i.e.*, (1) passes through $(-2, 0) \Rightarrow 4a^2 + b^2 = 16$

Using A.M. $\geq$ G.M.

$$\Rightarrow \frac{4a^2+b^2}{2}\geq\sqrt{4a^2\cdot b^2}$$

$$\Rightarrow ab\leq 4$$

71. (6) : We have $\frac{x-a}{\cos\theta}=\frac{y-2}{\sin\theta}=r$

$(a + r\cos\theta, 2 + r\sin\theta)$ lies on ellipse at A and D.

$$\frac{(a+r\cos\theta)^2}{9}+\frac{(2+r\sin\theta)^2}{4}=1$$

$$\Rightarrow r_1r_2=PA\cdot PD$$

PA, PB, PC and PD are in G.P

$\Rightarrow$ $PA\cdot PD = PB\cdot PC$

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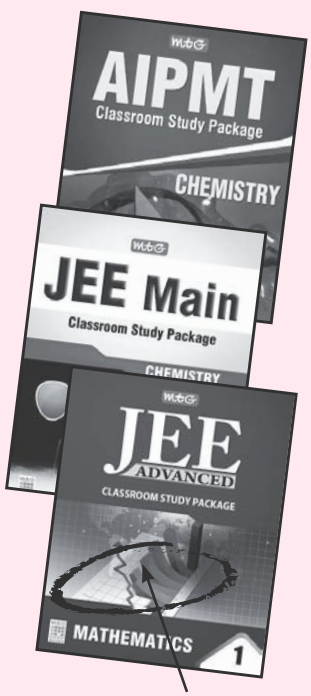

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concept BOOSTERS

Class XII

Trigonometry-II

* ALOK KUMAR, B.Tech, IIT Kanpur

This column is aimed at Class XII students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

SINE FORMULA

In any triangle ABC, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

COSINE FORMULA

- $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$ or $a^2 = b^2 + c^2 - 2bc\cos A$
- $\cos B = \frac{c^2 + a^2 - b^2}{2ca}$
- $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$

PROJECTION FORMULA

- $a = b\cos C + c\cos B$
- $b = c\cos A + a\cos C$
- $c = a\cos B + b\cos A$

NAPIER'S ANALOGY - TANGENT RULE

- $\tan\left(\frac{B-C}{2}\right) = \left(\frac{b-c}{b+c}\right)\cot\left(\frac{A}{2}\right)$
- $\tan\left(\frac{C-A}{2}\right) = \left(\frac{c-a}{c+a}\right)\cot\left(\frac{B}{2}\right)$
- $\tan\left(\frac{A-B}{2}\right) = \left(\frac{a-b}{a+b}\right)\cot\left(\frac{C}{2}\right)$

TRIGONOMETRIC FUNCTIONS OF HALF ANGLES

- $\cos\left(\frac{A}{2}\right) = \sqrt{\frac{s(s-a)}{bc}}$; $\cos\left(\frac{B}{2}\right) = \sqrt{\frac{s(s-b)}{ca}}$; $\cos\left(\frac{C}{2}\right) = \sqrt{\frac{s(s-c)}{ab}}$
- $\sin\left(\frac{A}{2}\right) = \sqrt{\frac{(s-b)(s-c)}{bc}}$; $\sin\left(\frac{B}{2}\right) = \sqrt{\frac{(s-c)(s-a)}{ca}}$; $\sin\left(\frac{C}{2}\right) = \sqrt{\frac{(s-a)(s-b)}{ab}}$
- $\tan\left(\frac{A}{2}\right) = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{\Delta}{s(s-a)}$

where $s = \frac{a+b+c}{2}$ & Δ = area of triangle.

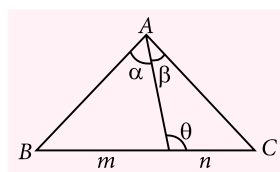
- Area of triangle = $\sqrt{s(s-a)(s-b)(s-c)}$.

m- n RULE

In any triangle,

$(m + n)\cot\theta = m\cot\alpha - n\cot\beta$

$= n\cot B - m\cot C$



- $\frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B$ = area of ΔABC.
- $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

* Alok Kumar is a winner of INDIAN NATIONAL MATHEMATICS OLYMPIAD (INMO-91). He trains IIT and Olympiad aspirants.

Note: $R=\dfrac{abc}{4\Delta}$; where R is the radius of circumcircle & Δ is area of triangle.

RADIUS OF THE INCIRCLE 'r'

- $r=(s-a)\tan\left(\dfrac{A}{2}\right)=(s-b)\tan\left(\dfrac{B}{2}\right)$
  $=(s-c)\tan\left(\dfrac{C}{2}\right)$
- $r=\dfrac{a\sin\left(\dfrac{B}{2}\right)\sin\left(\dfrac{C}{2}\right)}{\cos\left(\dfrac{A}{2}\right)}$ & so on
- $r=\dfrac{\Delta}{s}$, where $s=\dfrac{a+b+c}{2}$
- $r=4R\sin\left(\dfrac{A}{2}\right)\sin\left(\dfrac{B}{2}\right)\sin\left(\dfrac{C}{2}\right)$

RADIUS OF THE EX-CIRCLES r_1, r_2 & r_3

- $r_1=\dfrac{\Delta}{s-a}$; $r_2=\dfrac{\Delta}{s-b}$; $r_3=\dfrac{\Delta}{s-c}$
- $r_1=s\tan\left(\dfrac{A}{2}\right)$; $r_2=s\tan\left(\dfrac{B}{2}\right)$; $r_3=s\tan\left(\dfrac{C}{2}\right)$
- $r_1=4R\sin\left(\dfrac{A}{2}\right).\cos\left(\dfrac{B}{2}\right).\cos\left(\dfrac{C}{2}\right)$;
  $r_2=4R\sin\left(\dfrac{B}{2}\right).\cos\left(\dfrac{A}{2}\right).\cos\left(\dfrac{C}{2}\right)$;
  $r_3=4R\sin\left(\dfrac{C}{2}\right).\cos\left(\dfrac{A}{2}\right).\cos\left(\dfrac{B}{2}\right)$
- $r_1=\dfrac{a\cos\left(\dfrac{B}{2}\right)\cos\left(\dfrac{C}{2}\right)}{\cos\left(\dfrac{A}{2}\right)}$ & so on

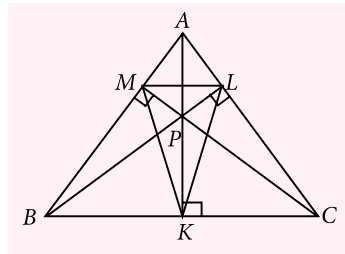
LENGTH OF ANGLE BISECTOR & MEDIANS

If m_a and β_a are the lengths of a median and an angle bisector from the angle A, then

$$m_a=\frac{1}{2}\sqrt{2b^2+2c^2-a^2} \text{ and } \beta_a=\frac{2bc\cos\left(\dfrac{A}{2}\right)}{b+c}$$

Note : $m_a^2+m_b^2+m_c^2=\dfrac{3}{4}(a^2+b^2+c^2)$.

ORTHOCENTRE AND PEDAL TRIANGLE



The triangle KLM which is formed by joining the feet of the altitudes is called the pedal triangle.

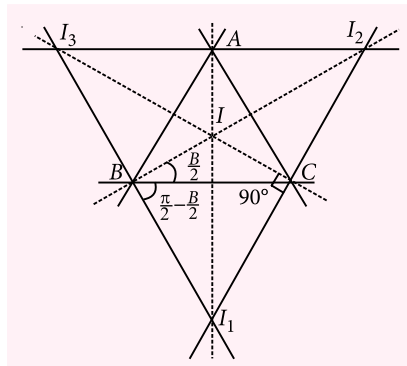
(i) The distances of the orthocentre from the angular points of the ΔABC are $2R\cos A$, $2R\cos B$ and $2R\cos C$.

(ii) The distances of P from the sides are $2R\cos B\cos C$, $2R\cos C\cos A$ and $2R\cos A\cos B$.

(iii) The sides of the pedal triangle are $a\cos A$ $(=R\sin 2A)$, $b\cos B$ $(=R\sin 2B)$ and $c\cos C$ $(=R\sin 2C)$ and its angles are $\pi-2A$, $\pi-2B$ and $\pi-2C$.

(iv) Circumradii of the triangles PBC, PCA, PAB and ABC are equal.

EXCENTRAL TRIANGLE



The triangle formed by joining the three excentres I_1, I_2 and I_3 of ΔABC is called the excentral or excentric triangle.

(i) Incentre I of ΔABC is the orthocentre of the excentral $\Delta I_1I_2I_3$.

(ii) ΔABC is the pedal triangle of the $\Delta I_1I_2I_3$.

(iii) The sides of the excentral triangle are

$$4R\cos\left(\frac{A}{2}\right),\ 4R\cos\left(\frac{B}{2}\right) \text{and } 4R\cos\left(\frac{C}{2}\right)$$

and its angles are

$\left(\frac{\pi}{2}-\frac{A}{2}\right), \left(\frac{\pi}{2}-\frac{B}{2}\right)$ and $\left(\frac{\pi}{2}-\frac{C}{2}\right)$.

(iv) $II_1 = 4R\sin\left(\frac{A}{2}\right)$; $II_2 = 4R\sin\left(\frac{B}{2}\right)$;

$II_3 = 4R\sin\left(\frac{C}{2}\right)$.

THE DISTANCES BETWEEN THE SPECIAL POINTS

- The distance between circumcentre and orthocentre of the ΔABC is
  $= R\sqrt{1-8\cos A\cos B\cos C}$.
- The distance between circumcentre and incentre of the ΔABC is $\sqrt{R^2-2Rr}$.
- The distance between incentre and orthocentre is $\sqrt{2r^2-4R^2\cos A\cos B\cos C}$
- Perimeter (P) and area (A) of a regular polygon of n sides inscribed in a circle of radius r are given by
  $P=2nr\sin\left(\frac{\pi}{n}\right)$ and $A=\frac{1}{2}nr^2\sin\left(\frac{2\pi}{n}\right)$
- Perimeter (P) and area (A) of a regular polygon of n sides circumscribed about a given circle of radius r are given by
  $P=2nr\tan\left(\frac{\pi}{n}\right)$ and $A=nr^2\tan\left(\frac{\pi}{n}\right)$

PROBLEMS

SECTION-I

Single Correct Answer Type

1. If median AD of a triangle ABC makes angle $\frac{\pi}{6}$ with side BC, then the value of $(\cot B-\cot C)^2$ is equal to

(a) 6 (b) 9 (c) 12 (d) 15

2. In a triangle with base $a = 10$ cm, if $b = 2c$. The maximum value of the altitude is

(a) $\frac{20}{3}$ cm (b) $\frac{16}{3}$ cm (c) 5 cm (d) 7 cm

3. In a ΔABC, if $\angle B+\angle C=\frac{2\pi}{3}$ and BI and CI be the angle bisectors of angles B and C respectively, then the value of $\frac{AP}{BI}+\frac{AQ}{CI}$, where P and Q are feet of perpendiculars from A on BI and CI respectively is

(a) $\frac{1}{\sqrt{3}}$ (b) $\frac{\sqrt{3}-1}{\sqrt{3}+1}$ (c) $\frac{\sqrt{3}+1}{\sqrt{3}-1}$ (d) $\sqrt{3}$

4. Three parallel chords of a circle have lengths 2, 3, 4 and subtend angles α, β, $\alpha+\beta$ at the centre respectively (given $\alpha+\beta<\pi$), then $\cos\alpha$ is equal to

(a) $\frac{15}{31}$ (b) $\frac{17}{35}$

(c) $\frac{17}{32}$ (d) none of these

5. If the perimeter of the triangle formed by foot of altitudes of the triangle ABC is equal to four times the circumradius of ΔABC, then ΔABC is

(a) isosceles triangle
(b) equilateral triangle
(c) right angled triangle
(d) none of these

6. The radius of the incircle of a triangle is 24 cm. The segments into which one side of a triangle is divided by the point of contact with incircle of the triangle are 36 cm and 48 cm, then the perimeter of the triangle is

(a) 146 cm (b) 150 cm
(c) 252 cm (d) none of these

7. In a triangle ABC, $\tan\left(\frac{A}{2}\right)=\frac{5}{6}$ and $\tan\left(\frac{C}{2}\right)=\frac{2}{5}$, then

(a) a, b, c are in A.P.
(b) $\cos A, \cos B, \cos C$ are in A.P.
(c) $\sin A, \sin B, \sin C$ are in G.P.
(d) none of these

8. In a ΔABC, sides a, b, c are in A.P., then $\sin B\sin C\cos^2(A/2)$, $\sin A\cos^2(B/2)\sin C$ and $\sin A\sin B\cos^2(C/2)$ are in

(a) G.P. (b) H.P.
(c) A.P. (d) none of these

9. In ΔABC, $A - B = 120°$ and $R = 8r$, then $\cos C$ is equal to

(a) $\frac{1}{8}$ (b) $\frac{7}{8}$ (c) $\frac{1}{4}$ (d) $\frac{5}{8}$

10. The acute angle of a rhombus whose side is geometric mean of its diagonals is

(a) 15° (b) 20° (c) 30° (d) 80°

11. In ΔABC, $AB = x$, $BC = x + 1$, $\angle C = \frac{\pi}{3}$, then the least integer value of x is

(a) 6 (b) 7

(c) 8 (d) none of these

12. A square is inscribed in an equilateral triangle of side 6 such that 2 vertices lie on one side, one each on other two sides except the square part. Thus there are 3 triangular parts. Sum of the circumradii of these three triangles is

(a) $9(2-\sqrt{3})$ (b) $18(2-\sqrt{3})$

(c) 6 (d) 9

13. In a ΔABC, medians AD and BE are drawn. If $AD = 4$, $\angle DAB = \frac{\pi}{6}$ and $\angle ABE = \frac{\pi}{3}$, then the area of ΔABC (in sq. units) is

(a) $\frac{64}{3}$ (b) $\frac{8}{3\sqrt{3}}$ (c) $\frac{16}{3}$ (d) $\frac{32}{3\sqrt{3}}$

14. If in a ΔABC, $A = \frac{\pi}{7}$, $B = \frac{2\pi}{7}$ and $C = \frac{4\pi}{7}$, then $a^2 + b^2 + c^2$ must be

(a) R^2 (b) $3R^2$ (c) $4R^2$ (d) $7R^2$

15. R is the circumradius of ΔABC whose circumcentre is 'S'. R' is the circumradius of ΔSBC. Then the ratio $R : R'$ is

(a) 1

(b) depends upon side BC

(c) independent of $\angle A$

(d) depends on $\angle A$

16. The sum of the infinite series $\cot^{-1}2 + \cot^{-1}8 + \cot^{-1}18 + \cot^{-1}32 + \ldots$ is equal to

(a) $\pi/3$ (b) $\pi/4$ (c) $\pi/6$ (d) $\pi/8$

17. The inequality $\sin^{-1}x > \cos^{-1}x$ holds for

(a) all values of x (b) $x \in (0, 1/\sqrt{2})$

(c) $x \in (1/\sqrt{2}, 1]$ (d) no value of x

18. If $\tan^{-1}\left(\frac{x+1}{x+2}\right) + \tan^{-1}\left(\frac{x-1}{x-2}\right) = \frac{\pi}{4}$, then $x =$

(a) ± 1 (b) $\pm\frac{1}{2}$ (c) $\pm\frac{1}{\sqrt{2}}$ (d) $\pm\frac{1}{2\sqrt{2}}$

19. If $\cot^{-1}x + \cot^{-1}y + \cot^{-1}z = \frac{\pi}{2}$, then $x + y + z$ is equal to

(a) $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$ (b) xyz

(c) $xy + yz + zx$ (d) none of these

20. $\sin^{-1}\frac{1}{\sqrt{2}} + \sin^{-1}\left(\frac{\sqrt{2}-\sqrt{1}}{\sqrt{6}}\right) + \ldots\ldots$

$\ldots\ldots + \sin^{-1}\left(\frac{\sqrt{n}-\sqrt{n-1}}{\sqrt{n(n+1)}}\right) + \ldots\ldots\ldots\ldots \infty =$

(a) 0 (b) $\pi/4$ (c) $\pi/2$ (d) $3\pi/4$

21. $\sum_{k=1}^{n} \tan^{-1}\left[\frac{2k}{2+k^2+k^4}\right] =$

(a) $\tan^{-1}(n^2 + n + 1)$

(b) $\tan^{-1}(n^2+n+1) - \frac{\pi}{4}$

(c) $\tan^{-1}(n^2 - n + 1) - \tan^{-1}(n^2 + n + 1)$

(d) none of these

22. The number of solutions of the equation $\left|\tan^{-1}|x|\right| = \sqrt{(x^2+1)^2 - 4x^2}$ is

(a) 2 (b) 3

(c) 4 (d) none of these

23. If $\sin^{-1}(\cos^{-1}x) < 1$ and $\cos^{-1}(\cos^{-1}x) < 1$, then $x \in$

(a) (sin 1, sin (sin 1))

(b) (cos (cos 1), cos(sin1))

(c) (cos(sin 1), cos(cos1))

(d) ϕ

24. The value of $\int_{-y}^{y} \left(\sec^{-1}x - \tan^{-1}\sqrt{x^2-1}\right)dx$;

$y > 1$ given that $\int_{1}^{y} \sec^{-1}x\,dx = \lambda$ is

(a) $\pi y - 2\lambda$ (b) $\pi(y - 1) - 2\lambda$

(c) $\pi(y - 1) - \lambda$ (d) none of these

25. If α, β, γ are the roots of the equation $x^3 + mx^2 + 3x + m = 0$, then the general value of $\tan^{-1}\alpha + \tan^{-1}\beta + \tan^{-1}\gamma$ is

(a) $(2n+1)\frac{\pi}{2}$ (b) $n\pi$

(c) $\frac{n\pi}{2}$ (d) $\frac{m}{3}$

26. Given that $0 \le x \le 1/2$, the value of

$$\tan\left[\sin^{-1}\left(\frac{x}{\sqrt{2}}+\sqrt{\frac{1-x^2}{2}}\right)-\sin^{-1}x\right] \text{ is}$$

(a) -1 (b) 1 (c) $1/\sqrt{3}$ (d) $\sqrt{3}$

SECTION-II

More than One Correct Answer Type

27. If Δ represents the area of the triangle ABC, then $\sqrt{a^2b^2-(2\Delta)^2}+\sqrt{b^2c^2-(2\Delta)^2}+\sqrt{c^2a^2-(2\Delta)^2}=$

(a) $a^2+b^2+c^2$

(b) $\dfrac{a^2+b^2+c^2}{2}$

(c) $ab\cos C + bc\cos A + ca\cos B$

(d) $ab\sin C + bc\sin A + ca\sin B$

28. If p, R, r be perimeter, circumradius and inradius of an arbitrary triangle, then which of the following may not be always true?

(a) $p > R + r$ (b) $p \le R + r$

(c) $\dfrac{p}{6} < R+r < 6p$ (d) none of these

29. If H is the orthocentre of triangle ABC, R = circumradius and $P = AH + BH + CH$, then

(a) $P = 2(R + r)$ (b) max. of P is $3R$

(c) min. of P is $3R$ (d) $P = 2(R - r)$

30. In any triangle ABC, $\dfrac{ab-r_1r_2}{r_3}$ is equal to

(a) $\dfrac{bc-r_2r_3}{r_1}$ (b) $\dfrac{ac-r_3r_1}{r_2}$

(c) 0 (d) None of these

31. In a ΔABC, a, c, A are given and b_1, b_2 are two values of the third side b is such that $b_2 = 2b_1$, then

(a) $\sin A = \sqrt{\dfrac{9a^2-c^2}{8a^2}}$ (b) $a > \dfrac{c}{3}$

(c) $\sqrt{\dfrac{9a^2-c^2}{8c^2}}$ (d) $\sin A = \sqrt{\dfrac{9a^2-c^2}{8c^2}}$

32. If the orthocentre of an isosceles triangle lies on the incircle of the triangle, then

(a) the base angle of the triangle is $\cos^{-1}\dfrac{2}{3}$.

(b) the triangle is acute angled.

(c) the base angle of the triangle is $\tan^{-1}\dfrac{\sqrt{5}}{2}$.

(d) If S, I are the circumcentre and incentre and R is circumradius, then $\dfrac{SI}{R}=\dfrac{1}{3}$.

33. There exists a triangle ABC satisfying the conditions

(a) $b\sin A = a, \angle A < \dfrac{\pi}{2}$ (b) $b\sin A > a, \angle A > \dfrac{\pi}{2}$

(c) $b\sin A > a, \angle A < \dfrac{\pi}{2}$ (d) $b\sin A < a, \angle A < \dfrac{\pi}{2}$

SECTION-III

Comprehension Type

Paragraph for Question No. 34 to 36

R is circumradii of ΔABC, H is orthocentre. R_1, R_2, R_3 are circumradii of ΔAHB, ΔAHC and ΔBHC. If AH produced meet the circumradii of ABC at M and intersect BC at L.

$\angle AHB = 180° - C$; $\dfrac{C}{\sin(180^o - C)} = 2R_1$;

$\dfrac{C}{\sin C} = 2R_1$ and $R_1 = R$. Then,

34. $R_1R_2 + R_2R_3 + R_1R_3$ is equal to

(a) $2R^2$ (b) $3R^2$ (c) $5R^2$ (d) R^2

35. Area of ΔAHB is

(a) $2R\cos A\cos B\cos C$ (b) $R^2\cos A\cos B\cos C$

(c) $2R^2\cos A\cos B\sin C$ (d) none of these

36. Ratio of area of ΔAHB to ΔBML is

(a) $\cos B : 2\cos A$ (b) $2 : 1$

(c) $\cos A : \cos B\cos C$ (d) none of these

Paragraph for Question No. 37 to 39

Let ABC be an acute triangle with $BC = a$, $CA = b$ and $AB = c$, where $a \ne b \ne c$. Let P is any point inside ΔABC and D, E, F denote foot of perpendiculars from 'P' onto the sides BC, CA and AB respectively. Now answer the following questions.

37. All positions of point 'P' for which ΔDEF is isosceles lie on

(a) the incircle of ΔABC
(b) lines of internal angle bisectors from A, B and C
(c) arcs of 3 circles
(d) none of these

38. Let $A(7, 0)$, $B(4, 4)$ and $C(0, 0)$ and ΔDEF is isosceles with $DE = DF$. Then the curve on which 'P' may lie
(a) $x = 4$ or $x + y = 7$ or $4x = 3y$
(b) $x = 4$ or $x^2 + y^2 = 4x + 4y$
(c) $3(x^2 + y^2) + 196 = 49\ (x + y)$
(d) none of these

39. If ΔDEF is equilateral, then 'P'
(a) Coincides with incentre of ΔABC
(b) Coincides with orthocentre of ΔABC
(c) Lies on pedal Δ of ABC
(d) none of these

Paragraph for Question No. 40 and 41

Consider the system of equations

$$\cos^{-1}x+(\sin^{-1}y)^2=\frac{p\pi^2}{4} \text{ and}$$

$$(\cos^{-1}x)(\sin^{-1}y)^2=\frac{\pi^2}{16},\ p\in Z$$

40. The value of p for which system of equations has a solution is
(a) 1 (b) 2 (c) 0 (d) –1

41. The value of 'x' which satisfies the system of equation is

(a) $\cos\left(\frac{\pi^2}{8}\right)$ (b) $\sin\left(\frac{\pi^2}{4}\right)$

(c) $\cos\left(\frac{\pi^2}{2}\right)$ (d) $\cos\left(\frac{\pi^2}{4}\right)$

Paragraph for Question No. 42 to 44

For $x\in\left(0,\frac{\pi}{4}\right)$,

$$\text{let } S_n=\sum_{r=1}^{2n}\sin(\sin^{-1}x^{3r-2}),C_n=\sum_{r=1}^{2n}\cos(\cos^{-1}x^{3r-1})$$

$$\text{and } T_n=\sum_{r=1}^{2n}\tan(\tan^{-1}x^{3r}),$$

where $n \in N$ and $n \geq 3$

42. The correct order of S_n, C_n and T_n is given by
(a) $S_n > T_n > C_n$ (b) $S_n < C_n < T_n$
(c) $S_n < T_n < C_n$ (d) $S_n > C_n > T_n$

43. The value of $\lim_{n\to\infty}(S_n+C_n+T_n)$ is equal to

(a) $\frac{1}{1-x}$ (b) $\frac{x}{1-x}$

(c) $\frac{1}{1-x}$ (d) $\frac{x}{1+x}$

44. The value of 'x' for which $S_n = C_n + T_n$, is

(a) $\sin\left(\frac{\pi}{5}\right)$ (b) $2\sin\left(\frac{\pi}{5}\right)$

(c) $2\sin\left(\frac{\pi}{10}\right)$ (d) $\sin\left(\frac{\pi}{10}\right)$

SECTION-IV

Matrix Match Type

45. If r_1, r_2, r_3 are in and ex-radii of ΔABC and r_1, r_2, r_3 are the roots of $x^3 - 11x^2 + 36x - 36 = 0$, then match the following.

Column-I		Column-II	
(A)	$\cos A + \cos B + \cos C =$	(p)	6
(B)	$\sin A + \sin B + \sin C =$	(q)	30
(C)	If I_1, I_2, I_3 are the excentres of ΔABC then area of $\Delta I_1I_2I_3$ (in sq. units) =	(r)	$\frac{7}{5}$
(D)	Area of ΔABC (in sq. units) =	(s)	$\frac{12}{5}$

46. In ΔABC, $AB = 9$, $AC = 17.5$, altitude from A to line BC cut at M, $AM = 3$. Then,

Column-I		Column-II	
(A)	$\tan \angle ABC =$	(p)	4.5
(B)	radius of circle which circumscribe ΔABC =	(q)	26.25
(C)	radius of circle which circumscribe ΔABM =	(r)	outside the ΔABC
(D)	orthocentre of ΔABC lie	(s)	$\frac{1}{2\sqrt{2}}$

47. In triangle ABC if $2a^2 + 4b^2 + c^2 = 4ab + 2ac$ where a, b, c are the sides of a triangle, then match the following.

Column-I		Column-II	
(A)	$\cos A =$	(p)	$\frac{1}{4}$
(B)	$\cos B =$	(q)	$\frac{7}{8}$
(C)	$\cos C =$	(r)	0
(D)	$\sin(A - C) =$	(s)	1

48. Match the following Column–I with Column–II in a strict sense of interval.

Column-I		Column-II	
(A)	If $x^2 - 4x - 2\pi \le \sin^{-1}(\sin 5)$ then 'x' lies in	(p)	$R - [-1, 1]$
(B)	If $f(x) = x^2 - 4$ and $g(x) = f(x) + 5$ such that $f(\lvert g(x)\rvert) > 0$, then 'x' lies in	(q)	$R - (-1, 1)$
(C)	If $3\sin\theta + 4\cos\theta = \frac{5(2a-3)}{2-3a}$ has a real solution for θ then 'a' lies in	(r)	$[-4, 4]$
(D)	If $4x^4 + 9y^4 = 64$ and $z = \sqrt{3}xy$, then 'z' lies in	(s)	$[-1, 5]$

SECTION-V

Integer Answer Type

49. In a triangle ABC, $a : b : c = 4 : 5 : 6$. The ratio of the radius of the circumcircle to that of the incircle is $\frac{p}{q}$, then $p - q =$

50. If $A = \frac{\pi}{7}; B = \frac{2\pi}{7}; C = \frac{4\pi}{7}$ then in ΔABC,

$$\frac{a^2 + b^2 + c^2}{R^2} =$$

51. In a triangle ABC, P and Q are the midpoints of AB and AC respectively. If O is the circumcentre of the triangle ABC, then the value of $\left(\frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta OPQ}\right)\cot B \cot C =$____

52. For $x, y, z, t \in R$ if $\sin^{-1}x + \cos^{-1}y + \sec^{-1}z \ge t^2 - \sqrt{2\pi}\,t + 3\pi$, then the value of $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z + \tan^{-1}\left(\sqrt{\frac{2}{\pi}}t\right)$ is

53. If $\cos^{-1}\left(\frac{x}{3\pi}\right) > 3\frac{\pi}{4}$, then the number of integral values of 'x' is

54. $K = \tan^{-1}\left(\frac{1}{2}\tan 2A\right) + \tan^{-1}(\cot A) + \tan^{-1}(\cot^3 A)$ and $\frac{\pi}{4} < A < \frac{\pi}{2}$ then $[K] =$

(where [.] denotes greatest integer function)

55. $\sin\left\{2\left(\sin^{-1}\frac{\sqrt{5}}{3} - \cos^{-1}\frac{\sqrt{5}}{3}\right)\right\}$ is equal to $\frac{k\sqrt{5}}{81}$, then $k =$

56. If α, β are the values of x satisfying $\tan^{-1}x + \tan^{-1}(1 - x) = \cot^{-1}\frac{7}{9}$ and $x \in (0, 1)$ then the value of $9(\alpha^2 + \beta^2)$ is

57. The number of integral values of 'x' for which $\sin^{-1}x > \cos^{-1}x$ is

SOLUTIONS

1. (c) : Applying m–n rule,

$$(BD + DC)\cot\frac{\pi}{6} = DC\cot B - BD\cot C$$

$\Rightarrow (\cot B - \cot C)^2 = 12$

2. (a) : $\Delta = \frac{1}{2}ap$, where p is the length of altitude.

$$\text{Also } p = \frac{bc\sin A}{a} = \frac{abc\sin A\sin(B + C)\sin(B - C)}{(b^2 - c^2)\sin^2 A}$$

$$= \frac{ab^2 r\sin(B - C)}{b^2(1 - r^2)}$$

$$= \frac{ar}{1-r^2}\sin(B-C) \le \frac{ar}{1-r^2} = \frac{10\times\frac{1}{2}}{1-\frac{1}{4}} = \frac{20}{3}\text{ cm}$$

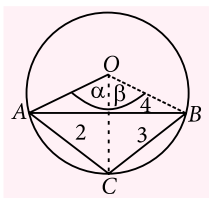
3. (d) : $\sin\left(\frac{B}{2}\right) = \frac{AP}{AB}$ and $\frac{BI}{\sin(A/2)} = \frac{AB}{\cos(C/2)}$

$$\Rightarrow \quad \frac{AP}{BI} = \frac{AB\sin(B/2)\cos(C/2)}{AB\sin(A/2)}$$

Similarly $\frac{AQ}{CI} = \frac{AC\sin(C/2)\cos(B/2)}{AC\sin(A/2)}$

$$\Rightarrow \quad \frac{AP}{BI} + \frac{AQ}{CI} = \cot\frac{A}{2} = \cot\frac{\pi}{6} = \sqrt{3}$$

4. (c) : Using the property that equal chords subtends equal angles at the centre of circle, then problem can be converted to the diagram in adjoining figure.



$AB = 4, AC = 2, BC = 3$

$\angle ABC = \alpha/2$

$$\cos(\alpha/2) = \frac{9+16-4}{2\times3\times4} = \frac{7}{8} \Rightarrow \cos\alpha = 2\cos^2(\alpha/2) - 1$$

$$= \left(2\times\frac{49}{64}\right) - 1 = \frac{98-64}{64} \Rightarrow \cos\alpha = \frac{34}{64} = \frac{17}{32}$$

5. (d) : $A'C = b\cos C$, $B'C = a\cos C$,

$A'B' = c\cos C$

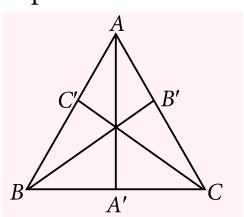
Similarly $A'C' = b\cos B$ and $B'C' = a\cos A$

Now $4R = a\cos A + b\cos B + c\cos C$

$\Rightarrow$ $\sin A\sin B\sin C = 1$

$\Rightarrow$ This is only possible when $\angle A = \angle B = \angle C = \pi/2$

So triangle is not possible.



6. (c) : $a = s - 36$, $b = s - 48$, $c = 84$, $r = 24$, now use $\Delta = rs$.

7. (a) : Here, $\tan\left(\frac{A}{2}\right)\tan\left(\frac{C}{2}\right) = \frac{s-b}{s}$

$$\Rightarrow \quad \frac{5}{6}\cdot\frac{2}{5} = \frac{s-b}{s} \Rightarrow 3s-3b = s \Rightarrow 2s = 3b$$

$\Rightarrow$ $a + b + c = 3b$ or $a + c = 2b$

$\Rightarrow$ a, b, c are in A.P.

8. (c) : a, b, c are in A.P. $\Rightarrow 2b = a + c$

$$\sin B\sin C\cos^2\left(\frac{A}{2}\right) = \frac{b}{2R}\cdot\frac{c}{2R}\cdot\frac{s(s-a)}{bc} = \frac{s(s-a)}{4R^2} \quad \ldots(1)$$

$$\text{Similarly, } \sin A\cos^2\left(\frac{B}{2}\right)\sin C = \frac{s(s-b)}{4R^2} \quad \ldots(2)$$

$$\sin A\sin B\cos^2\left(\frac{C}{2}\right) = \frac{s(s-c)}{4R^2} \quad \ldots(3)$$

Adding (1) and (3), we get

$$\frac{s(s-a)+s(s-c)}{4R^2} = \frac{s(2s-(a+c))}{4R^2}$$

$$= \frac{s(2s-2b)}{4R^2} \qquad [\because 2b = a+c]$$

$$= \frac{2s(s-b)}{4R^2} = 2\sin A\cos^2\left(\frac{B}{2}\right)\sin C$$

$\therefore$ $\sin B\sin C\cos^2\left(\frac{A}{2}\right), \sin A\cos^2\left(\frac{B}{2}\right)\sin C$

and $\sin A\sin B\cos^2\left(\frac{C}{2}\right)$ are in A.P.

9. (b) : $R = 8\times4R\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)$

$$\left(\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2}\right)\right)\sin\left(\frac{C}{2}\right) = 1/16$$

$$\Rightarrow \quad \sin\left(\frac{C}{2}\right) = \frac{1}{4} \text{ and } \cos C = \frac{7}{8}$$

10. (c) : If the acute angle is θ, then

$$\cos(\pi-\theta) = \frac{a^2+b^2-d^2}{2a^2}$$

$$(2a^2 - d_1^2) = -2a^2\cos\theta$$

$$\Rightarrow \quad d_1^2 = 4a^2\cos^2\frac{\theta}{2} \Rightarrow d_1 = 2a\cos\frac{\theta}{2}$$

Similarly, $d_2^2 = 2a^2(1-\cos\theta)$

$$\Rightarrow \quad d_2 = 2a\sin\frac{\theta}{2}$$

Now given $a^2 = d_1 d_2$

$\Rightarrow \quad a^2 = 4a^2 \sin\frac{\theta}{2}\cos\frac{\theta}{2} \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = 30°$

11. (b) : $x^2 = (x+1)^2 + b^2 - 2(x+1)b\cos\left(\frac{\pi}{3}\right)$

$0 = 2x + 1 + b^2 - (x + 1)\, b$

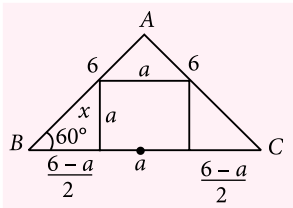
$b^2 - (x + 1)\, b + 2x + 1 = 0$

b is real $\Rightarrow \quad (x + 1)^2 - 4(2x + 1) \geq 0$

$x^2 - 6x - 3 \geq 0 \Rightarrow \quad x \geq 3 + \sqrt{12}$

The least integral value of x is 7.

12. (b) :



$$\tan 60° = \frac{a}{\left(\frac{6-a}{2}\right)} \Rightarrow \sqrt{3} = \frac{2a}{6-a}$$

$$\Rightarrow\ a = \frac{6\sqrt{3}}{2+\sqrt{3}} \ \Rightarrow\ a = 6\sqrt{3}(2-\sqrt{3})$$

$$\sin 60° = \frac{a}{x} \Rightarrow x = \frac{2a}{\sqrt{3}} \Rightarrow \frac{x}{2} = \frac{a}{\sqrt{3}} \ \text{(circumradius)}$$

The circumradii are $\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}$

$$\text{Sum} = 3 \times \frac{a}{\sqrt{3}} = \sqrt{3}a = \sqrt{3} \times 6\sqrt{3}(2-\sqrt{3}) = 18(2-\sqrt{3})$$

13. (d) : The medians intersect at centroid G with

$AG = \frac{8}{3}$ ($\because$ $AG : GD = 2 : 1$)

$$\angle AGB = \frac{\pi}{2} \Rightarrow BG = \frac{8}{3}\cot\left(\frac{\pi}{3}\right) = \frac{8}{3\sqrt{3}}$$

$$\text{Area of } \Delta AGB = \frac{1}{2} \times \frac{8}{3\sqrt{3}} \times \frac{8}{3} = \frac{32}{9\sqrt{3}}$$

$\therefore$ Area of $\Delta ABC = \frac{32}{3\sqrt{3}}$ sq. units

14. (d) : $A + B + C = \pi$

$4R^2(\sin^2 A + \sin^2 B + \sin^2 C)$

$= 2R^2(3 - (\cos 2A + \cos 2B + \cos 2C))$

$$= 2R^2\left(3 - \left(\frac{-1}{2}\right)\right) = 7R^2$$

15. (d) : $R = \frac{a}{\sin A}, R' = \frac{a}{\sin 2A}$

$\therefore \quad \frac{R}{R'} = 2\cos A$

16. (b) : The n^{th} term of the series can be written as

$u_n = \cot^{-1}(2n^2) = \tan^{-1}(1/2n^2)$

$$= \tan^{-1}\left(\frac{(2n+1)-(2n-1)}{1+(2n+1)(2n-1)}\right)$$

$= \tan^{-1}(2n + 1) - \tan^{-1}(2n - 1)$

so that $u_1 = \tan^{-1} 3 - \tan^{-1} 1$

$u_2 = \tan^{-1} 5 - \tan^{-1} 3$

$u_3 = \tan^{-1} 7 - \tan^{-1} 5$

$\vdots$

and the sum to n terms is therefore written as

$S_n = \tan^{-1}(2n + 1) - \tan^{-1} 1$

$\Rightarrow \quad \lim_{n\to\infty} S_n = \pi/2 - \pi/4 = \pi/4$

17. (c) : Given functions are defined for $-1 \leq x \leq 1$. If $-1 \leq x < 0$, $\sin^{-1} x < 0$ and $\cos^{-1} x > 0$ so there is no solution. When $0 \leq x \leq 1$, both functions have values in the interval $[0, \pi/2]$. Since the sine function increases monotonically in this interval.

$\sin(\sin^{-1} x) > \sin(\cos^{-1} x) \Rightarrow x > \sqrt{1 - x^2}$

$\Rightarrow \quad 2x^2 > 1$ or $x^2 > 1/2 \ \Rightarrow \ x \in (1/\sqrt{2}, 1]$

i.e. $\frac{1}{\sqrt{2}} < x \leq 1.$

18. (c) : If $A + B = \frac{\pi}{4}, (1 + \tan A)(1 + \tan B) = 2$

$$\Rightarrow \quad \frac{(2x-3)\cdot(2x+3)}{(x-2)(x+2)} = 2$$

$$\Rightarrow \quad 4x^2 - 9 = 2x^2 - 8 \Rightarrow x = \pm\frac{1}{\sqrt{2}}$$

19. (b) : $\cot^{-1} x + \cot^{-1} y + \cot^{-1} z = \frac{\pi}{2}$

$$\Rightarrow \quad \Sigma\left(\frac{\pi}{2} - \tan^{-1} x\right) = \frac{\pi}{2}$$

$$\Rightarrow \quad \Sigma \tan^{-1} x = \pi \quad \Rightarrow \quad \tan(\Sigma \tan^{-1} x) = 0$$

$$\Rightarrow \quad \Sigma \tan(\tan^{-1} x) = \Pi \tan(\tan^{-1} x)$$

$\Rightarrow \quad x + y + z = xyz$

20. (c) : $\lim_{n\to\infty}\left(\sum_{r=1}^{n}\sin^{-1}\left(\frac{\sqrt{r}-\sqrt{r-1}}{\sqrt{r(r+1)}}\right)\right)$

$= \lim_{n\to\infty}\left(\sum_{r=1}^{n}\sin^{-1}\left(\frac{1}{\sqrt{r}}\sqrt{1-\frac{1}{r+1}}-\frac{1}{\sqrt{r+1}}\sqrt{1-\frac{1}{r}}\right)\right)$

$= \lim_{n\to\infty}\sum_{r=1}^{n}\left(\sin^{-1}\frac{1}{\sqrt{r}}-\sin^{-1}\frac{1}{\sqrt{r+1}}\right)$

$= \lim_{n\to\infty}\left(\sin^{-1}1-\sin^{-1}\frac{1}{\sqrt{n+1}}\right)=\frac{\pi}{2}$

21. (b) : $T_n=\tan^{-1}\left[\frac{2n}{2+n^2+n^4}\right]$

$=\tan^{-1}\left[\frac{(n^2+n+1)-(n^2-n+1)}{1+(n^2+n+1)(n^2-n+1)}\right]$

$=\tan^{-1}(n^2+n+1)-\tan^{-1}(n^2-n+1)$

$\therefore\quad T_1=\tan^{-1}3-\tan^{-1}1$

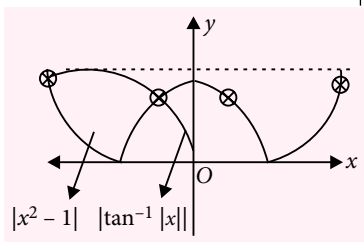
$T_2=\tan^{-1}7-\tan^{-1}3$

$T_3=\tan^{-1}13-\tan^{-1}7$ and so on

On adding, we get

$T_1+T_2+....+T_n=\tan^{-1}(n^2+n+1)-\frac{\pi}{4}$

22. (c) : $\sqrt{(x^2+1)^2-4x^2}=\sqrt{(x^2-1)^2}=|x^2-1|$



23. (c) : $\sin^{-1}(\cos^{-1}x)<1$

$0\le\cos^{-1}x\le\pi \quad \cos^{-1}x<\sin 1$

$0\le\cos^{-1}x\le\pi \quad x>\cos\sin 1$

$\cos 1\le x\le 1 \quad \therefore\quad x>\cos\sin 1$

$\cos^{-1}(\cos^{-1}x)<1$

$0\le\cos^{-1}x\le 1 \quad \cos^{-1}x>\cos 1$

$\cos 1\le x\le 1$

$\therefore\quad x>\cos\sin 1$ and $x<\cos\cos 1$

Also $\sin 1>\cos 1$

$\cos\sin 1<\cos\cos 1$

$\cos\sin 1<x<\cos\cos 1$

24. (b) : $\int_{-y}^{y}\left(\sec^{-1}x-\tan^{-1}\sqrt{x^2-1}\right)dx$

$=\int_{-y}^{-1}(\sec^{-1}x-(\pi-\sec^{-1}x))dx+\int_{1}^{y}(\sec^{-1}x-\sec^{-1}x)dx$

$=2\int_{-y}^{-1}\sec^{-1}x\,dx-\int_{-y}^{-1}\pi\,dx$

$=2\int_{1}^{y}(\pi-\sec^{-1}x)\,dx-\pi(y-1)=\pi(y-1)-2\lambda$

25. (b) : $\tan^{-1}\alpha+\tan^{-1}\beta+\tan^{-1}\gamma$

$=\tan^{-1}\left(\frac{\Sigma\alpha-\alpha\beta\gamma}{1-\Sigma\alpha\beta}\right)=\tan^{-1}(0)$

26. (b) : $0\le x\le 1/2\Rightarrow 0\le\sin y=x\le 1/2$

$\Rightarrow\quad 0\le y\le\pi/6$

$\sin^{-1}\left(\frac{1}{\sqrt{2}}\sin y+\frac{\cos y}{\sqrt{2}}\right)=\sin^{-1}\sin\left(y+\frac{\pi}{4}\right)$

If $0\le y\le\pi/6$

$\therefore\quad \tan\left[\sin^{-1}\left(\frac{x}{\sqrt{2}}+\sqrt{\frac{1-x^2}{2}}\right)-\sin^{-1}x\right]$

$=\tan^{-1}\left[y+\frac{\pi}{4}-y\right]=1$

27. (b, c) : Since, $2\Delta=bc\sin A=ca\sin B=ab\sin C$

$\Rightarrow$ Given expression

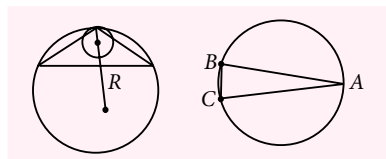
$=\sum\sqrt{a^2b^2-(ab\sin C)^2}=\sum ab\cos C$

$ab\cos C+bc\cos A+ca\cos B=\frac{a^2+b^2+c^2}{2}$

28. (a, b, c) : $R>p-r$

$R>2p-r>p-r$

$p>AB+AC>2R>R+r$



29. (a, b) : $AH=2R\cos A$, $BH=2R\cos B$,

$CH=2R\cos C$

$P=2R(\cos A+\cos B+\cos C)$

$= 2R\left(1+\frac{r}{R}\right) = 2(R+r)$

In any triangle $r \le \frac{R}{2}$

$\Rightarrow \quad P \le 2R + R \quad \therefore \quad P \le 3R$

30. (a, b) : Use $r_1 = \frac{\Delta}{s-a}, r_2 = \frac{\Delta}{s-b}, r_3 = \frac{\Delta}{s-c}$

31. (b, d) : $b^2 - 2bc\cos A + c^2 - a^2 = 0$

$\Rightarrow \quad b_1 + b_2 = 2c\cos A,\ b_1 b_2 = c^2 - a^2$

Now, $b_2 = 2b_1;\ b_1 = \frac{2c}{3}\cos A,$

$b_2 = \sqrt{\frac{c^2-a^2}{2}} \Rightarrow \frac{2c}{3}\cos A = \sqrt{\frac{c^2-a^2}{2}}$

$\Rightarrow \quad \sin A = \sqrt{\frac{9a^2-c^2}{8c^2}}$

32. (a, b, c, d) : Let ABC be the triangle in which $AB = AC$. Let I, P be the incentre and the orthocentre of the triangle respectively.

$AI = r\,\text{cosec}\frac{A}{2},\ AP = 2R\cos A$

$r\,\text{cosec}\frac{A}{2} = 2R\cos A + r$

33. (a, d) : The sine formula is $\frac{a}{\sin A} = \frac{b}{\sin B}$

$\Rightarrow \quad a\sin B = b\sin A$

$b\sin A = a \ \Rightarrow \ a\sin B = a \ \Rightarrow \ B = \pi/2$

$\therefore \quad \angle A < \frac{\pi}{2}$

$\therefore$ The triangle is possible.

$b\sin A > a \ \Rightarrow \ a\sin B > a \ \Rightarrow \ \sin B > 1$

$\therefore \quad \Delta ABC$ is not possible.

$b\sin A < a \ \Rightarrow \ a\sin B < a \ \Rightarrow \ \sin B < 1$

$\Rightarrow \quad \angle B$ does exist.

Now, if $b > a \ \Rightarrow \ \angle B > \angle A \quad \therefore \quad \angle A < \pi/2$

$\therefore$ The triangle is possible.

34. (b) : $R_1R_2 + R_2R_3 + R_1R_3 = 3R^2$

35. (c) : $AH = 2R\cos A$

$BH = 2R\cos B$

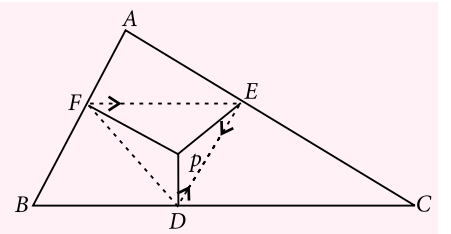
$\Delta = \frac{1}{2}(AH)(BH)\sin(180^o - C)$

$\Delta = 2R^2\cos A\cos B\sin C$

36. (c) : Ratio of area of ΔAHB to ΔBML is $2R^2\cos A\cos B\sin C : 2R^2\cos B\cos B\cos C\sin C\cos A : \cos B\cos C$

37. (c) : Let $DE = DF$ then $BDPE$ is cyclic with BP as diameter

$\Rightarrow \quad BP = \frac{DF}{\sin B} \Rightarrow DF = BP\sin B$ and $DE = CP\sin C$



But $DE = DF$

$\Rightarrow \quad \frac{BP}{CP} = \frac{\sin C}{\sin B} = \frac{c}{b}$

$\Rightarrow$ Locus of P is arcs of 3 circles.

38. (c) : $\frac{PB}{PC} = \frac{c}{b} = \frac{5}{7}$, Hence P will lie on circle (C)

39. (d) : For $DE = DF = EF$, P is a pointwhere the three arcs intersect.

40. (b) **41. (d)**

$\cos^{-1}x = \alpha,\ \alpha \in [0, \pi]$

$\sin^{-1}y = \beta,\ \beta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

$\alpha + \beta^2 = \frac{p\pi^2}{4}$ and $\alpha.\beta^2 = \frac{\pi^2}{16}$

Being $\beta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \Rightarrow \beta^2 \in \left[0, \frac{\pi^2}{4}\right]$

$\alpha + \beta^2 \in \left[0, \pi + \frac{\pi^2}{4}\right] \Rightarrow \frac{p\pi^2}{4} \in \left[0, \pi + \frac{\pi^2}{4}\right]$

$\Rightarrow \quad 0 \le p \le \frac{4}{\pi} + 1$

$\Rightarrow \quad p = 0, 1, 2\ (p \in Z)$

Take $\alpha\beta^2 = \frac{\pi^2}{16} \Rightarrow \alpha\left(\frac{p\pi^2}{4} - \alpha\right) = \frac{\pi^2}{16}$

$\Rightarrow \quad 16\alpha^2 - 4p\pi^2\alpha + \pi^4 = 0 \qquad \ldots(1)$

but $\alpha \in R$

$D \ge 0$

i.e., $16p^2\pi^2 - 64\pi^4 \ge 0 \ \Rightarrow \ p^2 \ge 4$

$p \geq 2 \quad \therefore \quad p = 2$

Put $p = 2$ in (1), we get

$$(4\alpha - \pi)^2 = 0 \Rightarrow \alpha = \frac{\pi^2}{4} = \cos^{-1} x$$

$$\therefore \quad x = \cos\left(\frac{\pi^2}{4}\right)$$

(42 - 44) :

We have $S_n = x + x^4 + x^7 + x^{10} + \ldots\ldots$ ($2n$ terms)

$C_n = x^2 + x^5 + x^8 + x^{11} + \ldots\ldots$ ($2n$ terms)

$T_n = x^3 + x^6 + x^9 + x^{12} + \ldots\ldots$ ($2n$ terms)

42. (d) : Clearly $S_n > C_n > T_n$ as 'x' is a proper fraction

$\therefore \quad x > x^2 > x^3$ and so on.

43. (b) : $\lim\limits_{n\to\infty}(S_n + C_n + T_n) = x + x^2 + x^3 + \ldots\ldots\infty = \dfrac{x}{1-x}$

44. (c) : We have $S_n = C_n + T_n$

$$\Rightarrow \quad x\frac{((x^3)^{2n}-1)}{(x^3-1)} = x^2\frac{((x^3)^{2n}-1)}{(x^3-1)} + x^3\frac{((x^3)^{2n}-1)}{(x^3-1)}$$

But $x \neq 1$, as $x \in \left(0, \dfrac{\pi}{4}\right)$, so, we get $x = x^2 + x^3$

$\Rightarrow \quad x^2 + x - 1 = 0 \ (x \neq 0)$

$$\Rightarrow \quad x = \frac{\sqrt{5}-1}{2} = 2\sin\frac{\pi}{10} \in \left(0, \frac{\pi}{4}\right)$$

45. A → r; B → s; C → q; D → p

$$\frac{1}{r} = \sum\frac{r_1 r_2}{r_1 r_2 r_3} = \frac{36}{36} = 1 \Rightarrow r = 1$$

$$R = \frac{1}{4}[r_1 + r_2 + r_3 - r] = \frac{1}{4}[11 - 1] = \frac{5}{2}$$

$$\cos A + \cos B + \cos C = 1 + \frac{r}{R} = 1 + \frac{2}{5} = \frac{7}{5}$$

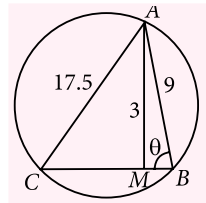
$$\sum r_1 r_2 = s^2 \Rightarrow s^2 = 36 \Rightarrow s = 6$$

$$\Delta = rs = 6$$

$$\sin A + \sin B + \sin C = \frac{2s}{2R} = \frac{6}{5/2} = \frac{12}{5}$$

$$\Delta I_1 I_2 I_3 = 2Rs = 5 \times 6 = 30$$

46. A → s; B → q; C → p; D → r



$$\sin\theta = \frac{1}{3},$$

$$\tan\theta = \frac{1}{2\sqrt{2}}$$

(B) $\dfrac{17.5}{\sin\theta} = 2R \Rightarrow R = 26.25$

(C) $\dfrac{3}{\sin\theta} = 2R_1 \Rightarrow R_1 = 4.5$

47. A → p; B → q; C → p; D → r

The given relation can be written as

$(a - 2b)^2 + (a - c)^2 = 0$

$$\Rightarrow \ a = 2b, a = c \ \Rightarrow \ b = \frac{a}{2}, c = a$$

$$\text{Now, } \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{\frac{a^2}{4} + a^2 - a^2}{2 \times \frac{a}{2} \times a} = \frac{1}{4}$$

$$\cos B = \frac{7}{8}, \ \cos C = \frac{1}{4}$$

$\because \quad A - C = 0 \ \Rightarrow \ \sin(A - C) = 0$

48. A → s; B → p; C → q; D → r

(A) $x^2 - 4x - 2\pi \leq \sin^{-1}\sin(5 - 2\pi)$

($5 - 2\pi$ is an angle in the 4th quadrant)

or $x^2 - 4x - 2\pi \leq 5 - 2\pi$

$\Rightarrow \ x^2 - 4x - 5 \leq 0 \ \Rightarrow \ (x - 5)(x + 1) \leq 0$

$\Rightarrow \ x \in [-1, 5]$

(B) $f\{g(x)\} = f(x^2 - 4 + 5) = f(x^2 + 1)$

$= (x^2 + 1)^2 - 4 = (x^2 + 3)(x^2 - 1)$

$\because \quad f\{g(x)\} > 0$

$\Rightarrow \ x^2 - 1 > 0 \quad \Rightarrow \quad x < -1 \text{ or } x > 1$

$\Rightarrow \ x \in R - [-1, 1]$

(C) $-5 \leq 3\sin\theta + 4\cos\theta \leq 5$

so, $-5 \leq \dfrac{5(2a-3)}{2-3a} \leq 5 \quad \therefore \quad -1 \leq \dfrac{2a-3}{2-3a} \leq 1$

$\Rightarrow \ a \in R - (-1, 1)$

(D) Let $x^2 = 4\cos\theta$ and $y^2 = \dfrac{8}{3}\sin\theta$, which satisfy

$4x^4 + 9y^4 = 64$

$$\therefore \quad z^2 = 3x^2y^2 = 3(4\cos\theta)\left(\frac{8}{3}\sin\theta\right) = 16\sin 2\theta \leq 16$$

$\therefore \quad z^2 - 16 \leq 0 \ \Rightarrow \ -4 \leq z \leq 4$

49. (9) : $a = 4k,\ b = 5k,\ c = 6k$

$$s = \frac{15}{2}k,\ s-a = \frac{7}{2}k,\ s-b = \frac{5}{2}k, s-c = \frac{3}{2}k$$

$$\Delta^2 = 15\times7\times5\times3\left(\frac{k}{2}\right)^4 \ \therefore\ \Delta = 15\sqrt{7}\left(\frac{k}{2}\right)^2$$

$$r = \frac{\Delta}{s} = 15\sqrt{7}\left(\frac{k}{2}\right)^2 \div \frac{15}{2}k = \sqrt{7}\,\frac{k}{2}$$

$$R = \frac{abc}{4\Delta} = \frac{4\cdot5\cdot6k^3}{4\cdot15\sqrt{7}\,k^2/4} = \frac{8}{\sqrt{7}}k$$

$$\therefore\quad \frac{R}{r} = \frac{8}{\sqrt{7}} \div \frac{\sqrt{7}}{2} = \frac{16}{7}$$

50. (7) : $a^2 + b^2 + c^2 =$
$2R^2\,(1 - \cos 2A + 1 - \cos 2B + 1 - \cos 2C)$

$$= 2R^2\left(3 - \left(\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{8\pi}{7}\right)\right)$$

$$= 2R^2\left(3 - \left(\frac{-1}{2}\right)\right) = 7R^2$$

51. (4) : Clearly $PQ = \frac{a}{2}, OA = R$ and $\angle AOQ = \angle B$

$OQ = OA\cos B = R\cos B$, now $\angle AOQ = \angle C$

Since, $PQ \parallel BC$

$$\Rightarrow\ \angle PQO = \frac{\pi}{2} - C$$

$$\Rightarrow\ \text{Area of } \Delta OPQ = \frac{1}{2}PQ\cdot OQ\sin\angle PQO$$

$$= \frac{aR}{4}\cos B\cos C$$

Now area of $\Delta ABC = \frac{1}{2}ab\sin C = aR\sin B\sin C$

$$\Rightarrow\ \frac{\text{Area of } ABC}{\text{Area of } OPQ}$$

$$= \frac{aR\sin B\sin C}{\frac{aR}{4}\cos B\cos C} = 4R\tan B\tan C$$

52. (0) : $\sin^{-1}x + \cos^{-1}y + \sec^{-1}z \le \frac{5\pi}{2}$

$$t^2 - \sqrt{2\pi}t + 3\pi = \left(t - \sqrt{\frac{\pi}{2}}\right)^2 + \frac{5\pi}{2} \ge \frac{5\pi}{2}$$

So solution exist if R.H.S. = L.H.S. $= \frac{5\pi}{2}$

i.e., $x = 1,\ y = -1,\ z = -1,\ t = \sqrt{\frac{\pi}{2}}$

53. (6) : $-1 \le \frac{x}{3\pi} \le \frac{-1}{\sqrt{2}} \ \Rightarrow\ -3\pi \le x \le \frac{-3\pi}{\sqrt{2}}$

$x = -9, -8, -7, -6, -5, -4$

54. (0) : $\frac{\pi}{4} < A < \frac{\pi}{2}$

$\Rightarrow\ \cot A < 1\ \Rightarrow\ \cot^4 A < 1$

$\therefore\ \tan^{-1}(\cot A) + \tan^{-1}(\cot^3 A)$

$$= \tan^{-1}\left(-\frac{1}{2}\tan 2A\right)$$

55. (8) : Let $\alpha = \sin^{-1}\frac{\sqrt{5}}{3}$, then $\cos^{-1}\frac{\sqrt{5}}{3} = \frac{\pi}{2} - \alpha$

Hence the expression $= \sin\left\{2\left(\alpha - \frac{\pi}{2} + \alpha\right)\right\}$

$= \sin(4\alpha - \pi) = -\sin 4\alpha$

$= -4\sin\alpha\cos\alpha(1 - 2\sin^2\alpha)$

$$= (-4)\left(\frac{\sqrt{5}}{3}\right)\left(\frac{2}{3}\right)\left(\frac{-1}{9}\right) = \frac{8\sqrt{5}}{81}$$

56. (5) : $0 < x < 1\ \Rightarrow\ 0 < 1 - x < 1$

$$\Rightarrow\ \tan^{-1}\left(\frac{1}{1-x+x^2}\right) = \tan^{-1}\frac{9}{7}\ \Rightarrow\ x = \frac{1}{3}, \frac{2}{3}$$

57. (1) : $\sin^{-1}x > \cos^{-1}x$ only for $x \in \left(\frac{1}{\sqrt{2}}, 1\right]$

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Indefinite Integration : Integration as the inverse process of differentiation, indefinite integrals of standard functions. Integration by the method of substitution, Integration by parts, Integration by partial fractions & special trigonometric functions.

Definite Integration : Definite Integrals and their properties, Fundamental theorem of integral calculus. Application of definite integrals to the determination of areas involving simple curves.

Differential Equations : Formation of ordinary differential equations, Solution of homogeneous differential equations, Separation of variables method, Linear first order differential equations.

PAPER-I

SECTION-I

Single Option Correct

This section contains 10 multiple choice questions. Each question has 4 choices, out of which ONLY ONE is correct.

1. $\int \cos\left\{2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right\}dx =$

(a) $\frac{1}{8}(x^2-1)+c$ (b) $\frac{1}{2}x^2+c$

(c) $\frac{1}{2}x+c$ (d) none of these

2. $\int x^x(1+\log x)dx =$

(a) $x^x \log x + c$ (b) $e^{x^x} + c$

(c) $x^x + c$ (d) none of these

3. $\int \frac{f(x)\cdot\phi'(x)-f'(x)\cdot\phi(x)}{f(x)\cdot\phi(x)}[\log\phi(x)-\log f(x)]dx =$

(a) $\log\left(\frac{\phi(x)}{f(x)}\right)+c$ (b) $\frac{1}{2}\left\{\log\frac{\phi(x)}{f(x)}\right\}^2+c$

(c) $\frac{\phi(x)}{f(x)}\log\left(\frac{\phi(x)}{f(x)}\right)+c$ (d) none of these

4. $f(x)=\int\frac{x^2dx}{(1+x^2)(1+\sqrt{1+x^2})}$ and $f(0)=0$, then $f(1)=$

(a) $\log(1+\sqrt{2})$ (b) $\log(1+\sqrt{2})-\frac{\pi}{4}$

(c) $\log(1+\sqrt{2})+\frac{\pi}{4}$ (d) none of these

5. $\int\frac{dx}{\cos x+\sqrt{3}\sin x}$ is equal to

(a) $\log\tan\left(\frac{x}{2}+\frac{\pi}{3}\right)+c$ (b) $\log\tan\left(\frac{x}{2}-\frac{\pi}{3}\right)+c$

(c) $\frac{1}{2}\log\tan\left(\frac{x}{2}+\frac{\pi}{3}\right)+c$ (d) none of these

6. If $a_n=\int_0^{\pi/4}\tan^n x\,dx$, then $a_2+a_4, a_3+a_5, a_4+a_6$ are in

(a) A.P. (b) G.P.

(c) H.P. (d) none of these

7. $\frac{d}{dx}F(x)=\frac{e^{\sin x}}{x}$, $x>0$.

If $\int_1^4\frac{2e^{\sin x^2}}{x}dx=F(k)-F(1)$, then one of the possible values of k is

(a) 4 (b) –4

(c) 16 (d) none of these

By : Sankar Ghosh, HOD(Math), Takshyashila. Mob : 09831244397

8. The value of $\int_0^{\pi/2} \sin^8 x\, dx =$

(a) $\frac{105\pi}{32(4!)}$ (b) $\frac{105\pi}{16(4!)}$

(c) $\frac{105}{16(4!)}$ (d) none of these

9. The general solution of the equation $(1+y^2)+(x-e^{\tan^{-1}y})\frac{dy}{dx}=0$ is

(a) $2xe^{\tan^{-1}y}=e^{2\tan^{-1}y}+c$

(b) $xe^{\tan^{-1}y}=\tan^{-1}y+c$

(c) $xe^{2\tan^{-1}y}=e^{\tan^{-1}y}+c$

(d) $x=2+ce^{-\tan^{-1}y}$

10. The solution of primitive integral equation $(x^2 + y^2)dy = xy\, dx$ is $y = y(x)$. If $y(1) = 1$ and $y(x_0) = e$ then x_0 is

(a) $\sqrt{2(e^2-1)}$ (b) $\sqrt{2(e^2+1)}$

(c) $\sqrt{3}e$ (d) $\sqrt{\frac{1}{2}(e^2+1)}$

SECTION-II

Multiple Correct Option

This section contains 5 multiple choice questions. Each question has 4 choices, out of which ONE or MORE is/ are correct.

11. If $A=\int_0^{\pi}\frac{\sin x}{\sin x+\cos x}dx$, $B=\int_0^{\pi}\frac{\sin x}{\sin x-\cos x}dx$, then

(a) $A + B = 0$ (b) $A = B$

(c) $A = B = \pi/2$ (d) $A = -B = \pi$

12. If $f(x)=\int_x^{x^2}\frac{dt}{(\log t)^2}$, $x\neq 0, x\neq 1$, then $f(x)$ is

(a) monotonically increasing in $(2, \infty)$

(b) monotonically increasing in $(1, 2)$

(c) monotonically decreasing in $(2, \infty)$

(d) monotonically decreasing in $(0, 1)$

13. If $\int x\log(1+x^2)dx=\phi(x)\cdot\log(1+x^2)+\psi(x)+c$, then

(a) $\phi(x)=\frac{1+x^2}{2}$ (b) $\psi(x)=\frac{1+x^2}{2}$

(c) $\psi(x)=-\frac{1+x^2}{2}$ (d) $\phi(x)=-\frac{1+x^2}{2}$

14. If $I_n=\int_0^{\pi/2}\cos^n x dx$, $n\in N$, then

(a) $I_{n-2} > I_n$ (b) $n(I_{n-2} - I_n) = I_{n-2}$

(c) $I_n : I_{n-1} = n : (n-1)$ (d) none of these

15. The area bounded by the curves $y = x^2$ and $x = y^2$ is equal to

(a) $\frac{1}{3}$ (b) $2\int_0^1 (x-x^2)dx$

(c) area of the region $\{(x, y) : x^2 \le y \le |x|\}$

(d) $\frac{3}{4}$

SECTION-III

Assertion & Reason Type

In each of the following questions two statements are given. Statement-1 (Assertion) and Statement-2 (Reason). Examine the statements carefully and answer the questions according to the instructions given below.

(a) If both Statement-1 and Statement-2 are correct and Statement-2 is the proper explanation of Statement-1.

(b) If both Statement-1 and Statement-2 are correct and Statement-2 is not the proper explanation of Statement-1.

(c) If Statement-1 is correct and Statement-2 is wrong.

(d) If Statement-1 is wrong and Statement-2 is correct.

16. **Statement-1 :** Area bounded by $y = e^x$, $y = 0$ and $x = 0$ is 1 square unit.

Statement-2 : Area bounded by $y = \log_e x$, $x = 0$ and $y = 0$ is 1 square unit.

17. $f(x)$ is a polynomial of degree 3 passing through the origin having local extrema at $x = \pm 2$.

Statement-1 : Ratio of the areas in which $f(x)$ cuts the circle $x^2 + y^2 = 36$ is 1 : 1.

Statement-2 : Both $y = f(x)$ and the circle are symmetrical about the origin.

18. Consider the differential equation

$$\frac{dy}{dx}=\frac{y}{2y\log y+y-x}$$

Statement-1 : $xy = y^2\log y + c$ is a solution of the given differential equation.

Statement-2 : The differential equation is a linear equation in y and x.

19. **Statement-1 :** $\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx,\ n \in N$

Statement-2 : $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

20. **Statement-1 :**

$$\int_{-4}^{-5} \sin(x^2-3)dx + \int_{-2}^{-1} \sin(x^2+12x+33)dx = 0.$$

Statement-2 : $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$

SECTION-IV

Integer Answer Type

This section contains 5 questions. The answer to each of the questions is a single-digit integer, ranging from 0 to 9. The appropriate bubbles below the respective question numbers in the ORS have to be darkened. For example, if the correct answers to question numbers X, Y, Z and W (say) are 6, 0, 9 and 2, respectively, then the correct darkening of bubbles will look like as given.

X	Y	Z	W
0	0	0	0
1	1	1	1
2	2	2	2
3	3	3	3
4	4	4	4
5	5	5	5
6	6	6	6
7	7	7	7
8	8	8	8
9	9	9	9

21. $\int 3^{2x}\cos 2x\,dx = \dfrac{k^x}{(\log k)^2+4}$

$[(\log k)\cos 2x + 2\sin 2x] + c$

then the value of k is

22. $\int \dfrac{dx}{4+5\sin^2 x} = \dfrac{1}{k}\tan^{-1}\left(\dfrac{3}{2}\tan x\right) + c,$

then the value of k is

23. If $I_n = \int_0^1 (1-x^5)^n\,dx$, then the value of $\dfrac{55}{8}\left(\dfrac{I_{10}}{I_{11}}\right)$ is equal to

24. If the curve satisfying

$(1+e^{x/y})dx + e^{x/y}\left(1-\dfrac{x}{y}\right)dy = 0$ passes through (1, 1) then $9 + y(2)e^{\frac{2}{y(2)}} - e$ is equal to

25. The area of the figure bounded by the curves $y = |x-1|$ and $y = 3 - |x|$ is

PAPER-II

SECTION-I

Single Option Correct

This section contains 10 multiple choice questions. Each question has 4 choices, out of which ONLY ONE is correct.

1. $\int \dfrac{\sin^8 x - \cos^8 x}{1-2\sin^2 x\cos^2 x}dx =$

(a) $\sin 2x + c$ (b) $-\dfrac{1}{2}\sin 2x + c$

(c) $\dfrac{1}{2}\sin 2x + c$ (d) $-\sin 2x + c$

2. If $\int \dfrac{dx}{5\cos^2 x + 4\sin x(\sin x + \cos x)}$

$= \dfrac{1}{\alpha}\tan^{-1}\left(\dfrac{1}{\beta} + \gamma\tan x\right) + k$, then α, β, γ are in

(a) A.P. (b) G.P.

(c) H.P. (d) A.G.P.

3. If $\int \dfrac{4e^x + 6e^{-x}}{9e^x - 4e^{-x}}dx = Ax + B\ln(9e^{2x} - 4) + c,$

then $A + B =$

(a) $-\dfrac{9}{35}$ (b) $-\dfrac{19}{36}$ (c) $\dfrac{19}{36}$ (d) $\dfrac{9}{35}$

4. Let $f(x) = \max\{2-x, 2, 1+x\}$ then $\int_{-1}^{1} f(x)\,dx =$

(a) 0 (b) 2 (c) $\dfrac{9}{2}$ (d) 3

5. For any $t \in R$ and f be a continuous function.

Let $I_1 = \int_{\sin^2 t}^{1+\cos^2 t} x \cdot f(x(2-x))\,dx$

and $I_2 = \int_{\sin^2 t}^{1+\cos^2 t} f(x(2-x))dx$ then $\dfrac{I_1}{I_2} =$

(a) 0 (b) 1 (c) 2 (d) 3

6. The order and degree of the differential equation $\sqrt{\frac{dy}{dx}} - 4\frac{dy}{dx} - 7x = 0$ are

(a) $1, \frac{1}{2}$ (b) 2, 1 (c) 1, 1 (d) 1, 2

7. The integrating factor of the differential equation $x \log x \frac{dy}{dx} + 2y = \log x$ is

(a) $\log x$ (b) $(\log x)^2$

(c) $\frac{1}{(\log x)}$ (d) x^2

8. The value of $\int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x} dx$ is equal to

(a) 0 (b) $\frac{\pi}{2}$ (c) $\frac{\pi^2}{4}$ (d) π^2

9. If the curve $y = a\sqrt{x} + bx$ passes through the point (1, 2) and the area bounded by the curve, the line $x = 4$ is 8 square unit, then the values of a and b are

(a) $a = 3, b = 1$ (b) $a = -3, b = 1$

(c) $a = 3, b = -1$ (d) $a = -3, b = -1$

10. The area (in square unit) of the smaller segment cut off from the circle $x^2 + y^2 = 9$ by line $x = 1$ is

(a) $\frac{1}{2}(9\sec^{-1}3 - \sqrt{8})$ (b) $(9\sec^{-1}3 - \sqrt{8})$

(c) $\sqrt{8} - 9\sec^{-1}3$ (d) $9\sec^{-1}3 + \sqrt{8}$

SECTION-II

Multiple Correct Option

This section contains 5 multiple choice questions. Each question has 4 choices, out of which ONE or MORE is/are correct.

11. Which of the following functions is/are homogeneous?

(a) $f(x, y) = \frac{x - y}{x^2 + y^2}$

(b) $f(x, y) = x^{1/3} y^{-2/3} \tan^{-1}\frac{x}{y}$

(c) $f(x, y) = x(\log\sqrt{x^2 + y^2} - \log y) + ye^{x/y}$

(d) $f(x, y) = x\left(\log\frac{2x^2 + y^2}{x} - \log(x + y)\right) + y^2 \tan\frac{x + 2y}{3x - y}$

12. The solution of the differential equation $\frac{dy}{dx} = \frac{x^2 + y^2 + 1}{2xy}$ satisfying $y(1) = 1$ is given by

(a) a system of hyperbola

(b) a system of circles

(c) $y^2 = x(1 + x) - 1$

(d) $(x - 2)^2 + (y - 3)^2 = 5$

13. The parabolas $y^2 = 4x$ and $x^2 = 4y$ divide the square region bounded by the lines $x = 4$, $y = 4$ and the coordinates axes. If S_1, S_2, S_3 are the areas of these parts numbered from top to bottom, respectively, then

(a) $S_1 : S_2 = 1 : 1$ (b) $S_2 : S_3 = 1 : 2$

(c) $S_1 : S_3 = 1 : 1$ (d) $S_1 : (S_1 + S_2) = 1 : 2$

14. $\int_0^{\pi/4} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x} =$

(a) $\frac{1}{ab}\tan^{-1}\left(\frac{b}{a}\right)$ if $a > 0, b > 0$

(b) $\frac{1}{ab}\tan^{-1}\left(\frac{b}{a}\right)$ if $a < 0, b < 0$

(c) $\frac{\pi}{4}$ if $a = 1, b = 1$ (d) none of these

15. $\int \frac{3x + 4}{x^3 - 2x - 4} dx = \log|x - 2| + \lambda\log[f(x)] + c$, then

(a) $\lambda = -\frac{1}{2}$ (b) $f(x) = x^2 + 2x + 2$

(c) $f(x) = |(x^2 + 2x + 2)|$ (d) $\lambda = \frac{1}{4}$

SECTION-III

Linked Comprehension Type

This section contains 4 paragraphs. Based upon each paragraph, 2 multiple choice questions have to be answered. Each question has 4 choices, out of which ONE or MORE is/are correct.

Paragraph for Question 16 and 17

Let $f(x) = \tan^{-1}\left(\frac{1 + \cos x}{\sin x}\right)$ and $g(x) = \tan^{-1}\left(\frac{\sin x}{1 - \cos x}\right)$

16. $\int\{f(x) + g(x)\}dx =$

(a) $\frac{\pi x}{2} - \frac{x^2}{4} + c$ (b) $\pi x - \frac{x^2}{2} + c$

(c) $\pi x + \frac{x^2}{4} + c$ (d) $\pi x + \frac{x^2}{2} + c$

17. $\int\{f(x)-g(x)\}\,dx =$

(a) $\pi x + c$ (b) $\frac{x^2}{2}+c$

(c) 0 (d) $\pi x + x^2 + c$

Paragraph for Question 18 and 19

$[x]$ denotes the greatest integer function.

18. $\int_0^{\pi} [\cot x]\,dx =$

(a) 1 (b) 0 (c) $-\frac{\pi}{2}$ (d) $\frac{\pi}{2}$

19. $\int_0^{50} \sin(x-[x])\,\pi dx =$

(a) 100π (b) 50π (c) $\frac{100}{\pi}$ (d) $\frac{50}{\pi}$

Paragraph for Question 20 and 21

An equation of the form $\frac{dy}{dx}+Py=Q$ is called Linear differential equation. Where P and Q are functions of x alone or constant. Taking $e^{\int P dx}$ as integrating factor, the above form reduces to $\frac{d}{dx}\left(ye^{\int P dx}\right)=Qe^{\int P dx}$

20. Solution of the equation $dx + xdy = e^{-y}\sec^2 ydy$ is

(a) $xe^y = \cot y + c$ (b) $xe^y = \tan y + c$

(c) $xe^y = x\tan y + c$ (d) $xe^y = x\cot y + c$

21. Solution of the equation $\cos t\frac{dx}{dt}+x\sin t=1$ is

(a) $x = c\sin t + \cos t$ (b) $x + c = \sin t + \cos t$

(c) $x = \sin t + c\cos t$ (d) none of these

Paragraph for Question 22 and 23

Consider the curves $C_1 : x = 0$; $C_2 : y = 0$; $C_3 : y = x^2 + 1$; $C_4 : y = 2$; $C_5 : x = 1$

22. The area enclosed by the curves C_1, C_2, C_3 and C_5 is

(a) $\frac{5}{6}$ (b) $\frac{4}{3}$ (c) $\frac{2}{3}$ (d) $\frac{7}{3}$

23. Area bounded by the curves C_3 and C_4 is (in square units)

(a) $\frac{4}{3}$ (b) $\frac{2}{3}$ (c) $\frac{5}{6}$ (d) $\frac{7}{5}$

SECTION-IV

Matrix-Match Type

This section contains 3 questions. Each question contains statements given in two columns, which have to be matched. The statements in Column-I are labelled as A, B, C and D, while the statements in Column-II are labelled as p, q, r and s. Any given statement in Column-I can have correct matching with ONE OR MORE statement(s) in Column-II. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following example. If the correct matches are A-p and s ; B-q and r; C-p and q; and D-s, then the correct darkening of bubbles will look like the above.

	p	q	r	s
A	(p)	(q)	(r)	(s)
B	(p)	(q)	(r)	(s)
C	(p)	(q)	(r)	(s)
D	(p)	(q)	(r)	(s)

24. Match the following:

Column I		Column II	
(A)	The area bounded by the curves $y = x\|x\|$, x-axis and the abscissa $x = -1$, $x = 1$ is	(p)	$\frac{10}{3}$
(B)	Area enclosed by the curves $y = x^2$ and $y = \|x\|$ is	(q)	$\frac{2}{3}$
(C)	The area bounded by $y^2 = x$, $y = 4$, $x = 0$ is	(r)	$\frac{1}{3}$
(D)	The area bounded by $x=\sqrt{y}, x=0$ and $x - y + 2 = 0$ is	(s)	$\frac{64}{3}$

25. Match the following:
The solution of

Column I		Column II	
(A)	$(x^4e^x - 4xy^2)dx + 4x^2 ydy = 0$ is	(p)	$x^2e^x + y = c$
(B)	$y(1 + xy)dx + x(1 - xy)dy = 0$ is	(q)	$e^x + 2\left(\frac{y}{x}\right)^2 = c$
(C)	$(3x - 2y + 1)dy = (6x - 4y + 3)dx$ is	(r)	$\log\frac{x}{y}-\frac{1}{xy}=c$
(D)	$(2 + x)xdx + e^{-x}dy = 0$	(s)	$4x - 2y - 2\log\lvert 3x - 2y + 3\rvert = c$

26. Match the following:

Column I		Column II	
(A)	$\int_{-1}^{1}[x+[x+[x]]]dx$	(p)	4
(B)	If $\int_0^{f(x)} t\cdot\cos t\,dt=\int_0^{x^2}(\sqrt{z}\cdot\sin\sqrt{z})dz$ and $f(0)=0$ then $4x\sin x+4\cos x-2x^2\cos x-f(x)\sin f(x)-\cos f(x)=$	(q)	1
(C)	$\int_0^{\pi/2}(\sin x)^{\cos x}\{\cos x\cot x-\log(\sin x)^{\sin x}\}dx$ is equal to	(r)	−3
(D)	If $f(x)$ is twice differentiable and $\lambda^2 f(x)-2\lambda f'(x)+f''(x)=0$ provides two equal values of λ for all x, and $f(0)=1, f'(0)=2$ then $f(\log_e 2)=$	(s)	3

SOLUTIONS

PAPER-I

1. (b): $\int\cos\left\{2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right\}dx$

$$=\int\frac{1-\tan^2\left(\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right)}{1+\tan^2\left(\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right)}dx$$

$$\left[\because\ \cos2\theta=\frac{1-\tan^2\theta}{1+\tan^2\theta}\right]$$

$$=\int\frac{1-\frac{1-x}{1+x}}{1+\frac{1-x}{1+x}}dx=\int x\,dx=\frac{x^2}{2}+c$$

2. (c): $\int x^x(1+\log x)dx=\int d(x^x)=x^x+c$

3. (b):

$$\int\frac{f(x)\cdot\phi'(x)-f'(x)\cdot\phi(x)}{f(x)\cdot\phi(x)}[\log\phi(x)-\log f(x)]\,dx$$

$$=\int\log\frac{\phi(x)}{f(x)}\cdot\frac{f(x)\cdot\phi'(x)-f'(x)\phi(x)}{f(x)\,\phi(x)}dx$$

$$=\int\log\frac{\phi(x)}{f(x)}\cdot d\left\{\log\frac{\phi(x)}{f(x)}\right\}=\frac{1}{2}\left\{\log\frac{\phi(x)}{f(x)}\right\}^2+c$$

$$\left[\because\ \int(f(x))^n f'(x)dx=\frac{(f(x))^{n+1}}{n+1}+C\right]$$

4. (b): $f(x)=\int\frac{x^2dx}{(1+x^2)(1+\sqrt{1+x^2})}$

$$=\int\frac{\tan^2\theta d\theta}{1+\sec\theta}\quad[\text{Putting } x=\tan\theta]$$

$$=\int\frac{\sin^2\theta d\theta}{\cos^2\theta\left(\frac{1+\cos\theta}{\cos\theta}\right)}=\int\frac{(1+\cos\theta)(1-\cos\theta)d\theta}{\cos\theta(1+\cos\theta)}$$

$$=\int\sec\theta d\theta-\int d\theta=\log|\sec\theta+\tan\theta|-\theta+c$$

$$f(x)=\log(x+\sqrt{1+x^2})-\tan^{-1}x+c\qquad\ldots\text{(i)}$$

$f(0)=0\ \Rightarrow\ c=0\ \ \therefore$ (i) becomes

$$f(x)=\log(x+\sqrt{1+x^2})-\tan^{-1}x$$

$$\Rightarrow\quad f(1)=\log(1+\sqrt{2})-\frac{\pi}{4}$$

5. (d): $\int\frac{dx}{\cos x+\sqrt{3}\sin x}=\frac{1}{2}\int\frac{dx}{\frac{1}{2}\cos x+\frac{\sqrt{3}}{2}\sin x}$

$$=\frac{1}{2}\int\frac{dx}{\cos\left(x-\frac{\pi}{3}\right)}=\frac{1}{2}\int\sec\left(x-\frac{\pi}{3}\right)dx$$

$$=\frac{1}{2}\log\left[\sec\left(x-\frac{\pi}{3}\right)+\tan\left(x-\frac{\pi}{3}\right)\right]+c$$

$$=\frac{1}{2}\log\tan\left(\frac{\pi}{4}+\frac{x-\frac{\pi}{3}}{2}\right)+c$$

$$\left[\sec\theta+\tan\theta=\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right]$$

$$=\frac{1}{2}\log\tan\left(\frac{x}{2}+\frac{\pi}{12}\right)+c$$

6. (c): $a_n=\int_0^{\pi/4}\tan^n x\,dx$

$$= \int_0^{\pi/4} \tan^{n-2} x(\sec^2 x - 1)dx$$

$$= \int_0^{\pi/4} \tan^{n-2} x \; d(\tan x) - a_{n-2}$$

$$= \left[\frac{\tan^{n-1} x}{n-1}\right]_0^{\pi/4} - a_{n-2}$$

$$a_n = \frac{1}{n-1} - a_{n-2} \Rightarrow a_n + a_{n-2} = \frac{1}{n-1}$$

$$\therefore \; a_4 + a_2 = \frac{1}{3},\; a_5 + a_3 = \frac{1}{4},\; a_6 + a_4 = \frac{1}{5}$$

Thus $a_4 + a_2$, $a_5 + a_3$, $a_6 + a_4$ are in H.P.

7. (c) : $\int_1^4 \frac{2e^{\sin x^2}}{x} dx$ $\left[\text{Putting } x^2 = z \;\; 2xdx = dz;\; 2dx = \frac{dz}{x}\right]$

$$= \int_1^{16} \frac{e^{\sin z}}{z} dz$$

x	1	4
z	1	16

$$= \int_1^{16} d(F(z)) \left[\because \frac{d}{dx}F(x) = \frac{e^{\sin x}}{x},\; x > 0\right]$$

$$= [F(z)]_1^{16} = F(16) - F(1)$$

$\therefore$ $F(16) = F(k)$, from the question.

Hence, one of the possible values of $k = 16$.

8. (a) : $I_8 = \int_0^{\pi/2} \sin^8 x \, dx$

Let $I_n = \int_0^{\pi/2} \sin^n x \, dx = \int_0^{\pi/2} \sin^{n-1} x \sin x \, dx$

$$= \left[\sin^{n-1} x.(-\cos x)\right]_0^{\pi/2} - \int_0^{\pi/2} ((-\cos x)(n-1)\sin^{n-2} x.\cos x)\, dx$$

$$= (n-1)\int_0^{\pi/2} \sin^{n-2} x \; \cos^2 x dx$$

$$= (n-1)\int_0^{\pi/2} (\sin^{n-2} x - \sin^n x)dx$$

$$I_n = (n-1)I_{n-2} - (n-1)I_n$$

$$\therefore \quad nI_n = (n-1)I_{n-2} \;\therefore\; I_n = \frac{n-1}{n} I_{n-2}$$

$$\therefore \quad I_8 = \frac{7}{8}I_6;\; I_6 = \frac{5}{6}I_4;\; I_4 = \frac{3}{4}I_2$$

$$I_2 = \frac{1}{2}I_0, \text{ where } I_0 = \int_0^{\pi/2} \sin^0 x dx = \frac{\pi}{2}$$

$$\therefore \quad I_8 = \frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{1\cdot3\cdot5\cdot7}{1\cdot2\cdot3\cdot4}\cdot\frac{1}{2^4}\cdot\frac{\pi}{2} = \frac{105\pi}{32(4!)}$$

9. (a) : $(1+y^2) + (x - e^{\tan^{-1} y})\frac{dy}{dx} = 0$

$$\Rightarrow (1+y^2) = (e^{\tan^{-1} y} - x)\frac{dy}{dx}$$

$$\Rightarrow \frac{1+y^2}{e^{\tan^{-1} y} - x} = \frac{dy}{dx} \Rightarrow \frac{dx}{dy} = \frac{e^{\tan^{-1} y} - x}{1+y^2}$$

$$\Rightarrow \frac{dx}{dy} + \frac{1}{1+y^2}x = \frac{e^{\tan^{-1} y}}{1+y^2}$$

[which is in the linear form]

$$\text{I.F.} = e^{\int \frac{dy}{1+y^2}} = e^{\tan^{-1} y}$$

So, $xe^{\tan^{-1} y} = \int \frac{(e^{\tan^{-1} y})^2}{1+y^2} dy$ [Putting $e^{\tan^{-1} y} = z$]

$$\Rightarrow \quad xe^{\tan^{-1} y} = \int z dz = \frac{1}{2}(e^{\tan^{-1} y})^2 + \frac{c}{2}$$

$$\Rightarrow \quad 2xe^{\tan^{-1} y} = e^{2\tan^{-1} y} + c$$

10. (c) : $(x^2 + y^2)dy = xy \, dx$

$$\Rightarrow \frac{dy}{dx} = \frac{xy}{x^2+y^2} \quad (\text{Put } y = vx)$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{v}{1+v^2}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v}{1+v^2} - v = \frac{v - v - v^3}{1+v^2} = \frac{-v^3}{1+v^2}$$

$$\Rightarrow \frac{1+v^2}{v^3}dv = -\frac{dx}{x}$$

$$\Rightarrow -\frac{1}{2v^2} + \log|v| = -\log|x| - \log c$$

$$\Rightarrow \log|v| + \log|x| + \log c = \frac{1}{2v^2}$$

$$\Rightarrow \log\left|\frac{y}{x}\cdot x\cdot c\right| = \frac{x^2}{2y^2}$$

$\Rightarrow \log|yc| = \frac{x^2}{2y^2} \quad \therefore \quad yc = e^{\frac{x^2}{2y^2}}$

$\therefore$ When $y(1) = 1$ then $c = e^{1/2}$

$\therefore \log y + \log e^{1/2} = \frac{x^2}{2y^2} \Rightarrow \log y - \frac{x^2}{2y^2} = -\frac{1}{2}$

For $x = x_0$, $y = e \Rightarrow 1 - \frac{x_0^2}{2e^2} = -\frac{1}{2}$

$\Rightarrow \frac{x_0^2}{2e^2} = 1 + \frac{1}{2} = \frac{3}{2} \Rightarrow x_0^2 = 3e^2$

$\therefore \quad x_0 = \sqrt{3}e$

11. (b, c) : $A = \int_0^{\pi} \frac{\sin x dx}{\sin x + \cos x}$

$= \int_0^{\pi} \frac{\sin(\pi - x)dx}{\sin(\pi - x) + \cos(\pi - x)}$

$= \int_0^{\pi} \frac{\sin x dx}{\sin x - \cos x} = B$

$\therefore \quad A = B$

Now, $A + B = \int_0^{\pi} \left[\frac{\sin x}{\sin x + \cos x} + \frac{\sin x}{\sin x - \cos x}\right] dx$

$= \int_0^{\pi} \frac{2\sin^2 x}{\sin^2 x - \cos^2 x} dx = \int_0^{\pi} \frac{1 - \cos 2x}{-\cos 2x} dx$

$= \int_0^{\pi} (1 - \sec 2x)dx = \pi - \left[\frac{\log(\sec 2x + \tan 2x)}{2}\right]_0^{\pi}$

$= \pi$. Hence $A = B = \pi/2$

12. (a, d) : $f(x) = \int_x^{x^2} \frac{dt}{(\log t)^2}, \; x \neq 0, \; x \neq 1$

$f(x) = \int_0^{x^2} \frac{dt}{(\log t)^2} - \int_0^{x} \frac{dt}{(\log t)^2}$

$\therefore \quad f'(x) = \frac{1}{(\log x^2)^2} \cdot 2x - \frac{1}{(\log x)^2} \cdot 1$

$f'(x) = \frac{2x}{4(\log x)^2} - \frac{1}{(\log x)^2} = \left(\frac{x}{2} - 1\right)\frac{1}{(\log x)^2}$

$\therefore$ In $(2, \infty)$, $f'(x) > 0$, so $f(x)$ is monotonically increasing in $(2, \infty)$

Again in $(0, 1)$, $f'(x) < 0$, so $f(x)$ is monotonically decreasing in $(0, 1)$.

13. (a, c) : $\int x \log(1 + x^2)\, dx$

$= \frac{x^2}{2}\log(1 + x^2) - \int \left(\frac{2x}{1 + x^2} \cdot \frac{x^2}{2}\right) dx$

$= \frac{x^2}{2}\log(1 + x^2) - \int \frac{x^3}{1 + x^2} dx$

$= \frac{x^2}{2}\log(1 + x^2) - \int \left(x - \frac{x}{1 + x^2}\right) dx$

$= \frac{x^2}{2}\log(1 + x^2) - \frac{x^2}{2} + \frac{1}{2}\log(1 + x^2) + c_1$

$= \frac{x^2 + 1}{2}\log(1 + x^2) - \frac{x^2 + 1}{2} + c$

$\therefore \quad \phi(x) = \frac{x^2 + 1}{2}, \; \psi(x) = -\frac{x^2 + 1}{2}$

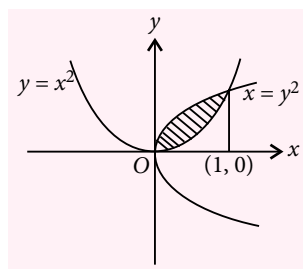
14. (a, b) : As in Q. No. 8,

$I_n = \frac{n-1}{n} I_{n-2}$

and I_n, I_{n-1} are +ve because $\cos^n x \geq 0$ in $[0, \pi/2]$. Therefore, $I_{n-2} > I_n$.

Also $I_n = \frac{n-1}{n} I_{n-2}$

15. (a, b, c) :



The equation of the curves are $y = x^2$ and $x = y^2$

Enclosed area $= \int_0^1 x^{1/2} dx - \int_0^1 x^2 dx = \frac{1}{3}$ sq. units

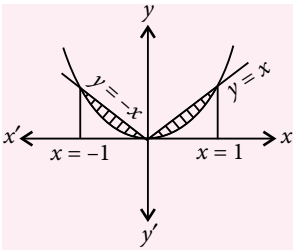
Thus option (a) is true.

Now, $2\int_0^1 (x - x^2)dx = 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 2\left(\frac{1}{6}\right) = \frac{1}{3}$

So, option (b) is correct.

Now, the area of the region bounded by $\{(x, y) : x^2 \leq y \leq |x|\}$ is

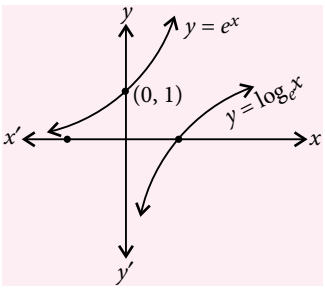
$2 \cdot \int_0^1 (x - x^2)dx = \frac{1}{3}$



16. (a) : Area bounded by $y = e^x, y = 0$ and $x = 0$ is

$$\int_{-\infty}^{0} e^x dx = e^x\Big|_{-\infty}^{0}$$

$= 1$ square unit



Again area bounded by $y = \log_e x$, $x = 0$ and $y = 0$ is

$$\int_{-\infty}^{0} x\,dy = \int_{-\infty}^{0} e^y dy$$

$= 1$ square unit

17. (a) : $y = f(x)$ is a polynomial of degree 3 which indicate that graph of $f(x)$ is symmetrical about origin, also the circle is symmetrical about origin. So the function $f(x)$ divide the area of circle into two equal parts.

18. (a) : $\dfrac{dy}{dx} = \dfrac{y}{2y\log y + y - x}$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{2y\log y + y - x}{y}$$

or $\quad \dfrac{dx}{dy} + \dfrac{1}{y}x = 2\log y + 1$

Here, $I.F. = e^{\int \frac{1}{y}dy} = e^{\log_e y} = y$

$$\Rightarrow \quad xy = \int (2y\log y + y)dy$$

$$= 2\left(\frac{y^2}{2}\log y - \int \frac{1}{y}\cdot\frac{y^2}{2}dy\right) + \frac{y^2}{2} + c$$

$$= y^2\log y - \frac{y^2}{2} + \frac{y^2}{2} + c$$

or $\quad xy = y^2\log y + c$

19. (a) : See Q. No. 8 and 11

20. (c)

21. (9) : $\int 3^{2x}\cos 2x dx = \int e^{\log 3^{2x}}\cos 2x dx$

$$= \int e^{(2\log 3)x}\cos 2x dx \quad = \int e^{(\log 9)x}\cos 2x\,dx$$

$$= \frac{e^{(\log 9)x}}{(\log 9)^2 + (2)^2}[(\log 9)\cos 2x + 2\sin 2x] + c$$

$\therefore$ By the question, $k = 9$.

22. (6) : $\int \dfrac{dx}{4 + 5\sin^2 x}$

$$= \int \frac{2dx}{8 + 5(1 - \cos 2x)}$$

$$= 2\int \frac{dx}{13 - 5\cos 2x}$$

$$= 2\int \frac{\frac{dz}{1+z^2}}{13 - 5\cdot\frac{1-z^2}{1+z^2}}$$

Let $\tan x = z$

$\sec^2 x dx = dz$

$dx = \dfrac{dz}{\sec^2 x} = \dfrac{dz}{1 + \tan^2 x}$

$dx = \dfrac{dz}{1+z^2}$

and $\cos 2x = \dfrac{1-z^2}{1+z^2}$

$$= 2\int \frac{dz}{13(1+z^2) - 5(1-z^2)}$$

$$= 2\int \frac{dz}{18z^2 + 8} = \int \frac{dz}{(3z)^2 + (2)^2}$$

$$= \frac{1}{2}\cdot\frac{1}{3}\tan^{-1}\left(\frac{3}{2}\tan x\right) + c$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{3}{2}\tan x\right) + c$$

$\therefore$ By the question, $k = 6$.

23. (7) : $I_n = \int_0^1 (1 - x^5)^n dx$

$$\therefore \quad I_{11} = \int_0^1 (1-x^5)^{11} dx = \int_0^1 (1-x^5)(1-x^5)^{10} dx$$

$$I_{11} = \int_0^1 (1-x^5)^{10} dx - \int_0^1 x^5(1-x^5)^{10} dx \qquad \text{... (i)}$$

Now, $\int x^5(1-x^5)^{10} dx = \int x(1-x^5)^{10}\cdot x^4 dx$

$$= \int x(1-x^5)^{10}\, d\left(\frac{1-x^5}{-5}\right)$$

$$= x\int (1-x^5)^{10} d\left(\frac{1-x^5}{-5}\right) - \int\left\{1\cdot\int (1-x^5)^{10} d\left(\frac{1-x^5}{-5}\right)\right\}dx$$

$$= -\frac{x}{5}\frac{(1-x^5)^{11}}{11} - \int\left\{-\frac{1}{5}\frac{(1-x^5)^{11}}{11}\right\}dx$$

$$\therefore \int x^5(1-x^5)^{10}dx = -\frac{x}{55}(1-x^5)^{11} + \frac{1}{55}\int(1-x^5)^{11}dx + c$$

Now (i) becomes

$$I_{11} = I_{10} + \frac{x(1-x^5)^{11}}{55}\Bigg|_0^1 - \frac{1}{55}I_{11}$$

$$\left(1+\frac{1}{55}\right)I_{11} = I_{10} \Rightarrow \frac{56}{55}I_{11} = I_{10}$$

$$\therefore \frac{I_{10}}{I_{11}} = \frac{56}{55}. \text{ So } \frac{55}{8}\left(\frac{I_{10}}{I_{11}}\right) = \frac{55}{8}\times\frac{56}{55} = 7$$

24. (8) : $(1+e^{x/y})dx + e^{x/y}\left(1-\frac{x}{y}\right)dy = 0$

$$\Rightarrow \frac{dx}{dy} = \frac{-e^{x/y}\left(1-\frac{x}{y}\right)}{1+e^{x/y}}$$

$$\Rightarrow v + y\frac{dv}{dy} = \frac{-e^v(1-v)}{1+e^v} \text{ [Putting } x = vy\text{]}$$

$$\Rightarrow y\frac{dv}{dy} = -\frac{v+e^v}{1+e^v} \Rightarrow \int\frac{1+e^v}{v+e^v}dv = -\int\frac{dy}{y}$$

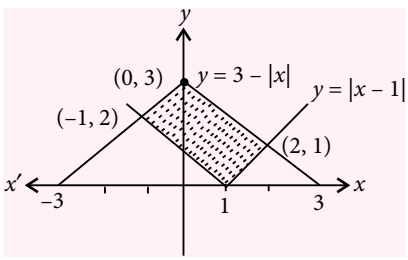
$\Rightarrow \log(v + e^v) + \log y = \log c$

$\Rightarrow x + ye^{x/y} = c$ ... (i)

When (i) passes through (1, 1), then $c = 1 + e$

But $2 + y(2)e^{\frac{2}{y(2)}} = 1 + e \therefore 9 + y(2)e^{\frac{2}{y(2)}} - e = 8$

25. (4) :



The area enclosed by the curve $y = |x - 1|$ and $y = 3 - |x|$ is shown in the adjacent diagram.

The curves intersect at the points (2, 1) and (–1, 2) forming a rectangle of $\sqrt{2}$ and $2\sqrt{2}$

$\therefore$ Required area = $\sqrt{2}\times 2\sqrt{2} = 4$

PAPER-II

1. (b) : $\int\frac{\sin^8 x - \cos^8 x}{1-2\sin^2 x\cos^2 x}dx$

$$= \int\frac{(\sin^4 x + \cos^4 x)(\sin^2 x - \cos^2 x)}{(1-2\sin^2 x\cos^2 x)}dx$$

$$= \int\frac{(\sin^4 x + \cos^4 x)(-\cos 2x)}{1-2\sin^2 x\cos^2 x}dx$$

$$= -\int\frac{(1-2\sin^2 x\ \cos^2 x)\cos 2x}{(1-2\sin^2 x\cos^2 x)}dx$$

$$= -\int\cos 2x\,dx = -\frac{1}{2}\sin 2x + c$$

2. (b) : $\int\frac{dx}{5\cos^2 x + 4\sin x(\sin x + \cos x)}$

$$= \int\frac{\sec^2 x\,dx}{5 + 4\tan^2 x + 4\tan x}$$

$$= \int\frac{dz}{4z^2+4z+5} \quad [\text{Putting } \tan x = z]$$

$$= \frac{1}{4}\int\frac{dz}{z^2+z+\frac{5}{4}} = \frac{1}{4}\int\frac{dz}{\left(z+\frac{1}{2}\right)^2+1}$$

$$= \frac{1}{4}\tan^{-1}\left(z+\frac{1}{2}\right) + c = \frac{1}{4}\tan^{-1}\left(\frac{1}{2}+\tan x\right) + c$$

$\therefore$ by the question, we have

$\alpha = 4, \beta = 2, \gamma = 1$

$\therefore$ α, β, γ are in G.P.

3. (b) : $\int\frac{4e^x + 6e^{-x}}{9e^x - 4e^{-x}}dx$, Putting $u = e^x$

$$= \int\frac{4u + 6u^{-1}}{9u - 4u^{-1}}\cdot\frac{du}{u}$$

$$= \int\frac{4u^2+6}{u(9u^2-4)}du = \int\frac{4u^2+6}{u(3u+2)(3u-2)}du$$

Let $\frac{4u^2+6}{u(3u+2)(3u-2)} = \frac{A}{u} + \frac{B}{3u+2} + \frac{C}{3u-2}$

$\Rightarrow 4u^2 + 6 = A(3u+2)(3u-2) + Bu(3u-2) + Cu(3u+2)$

Putting $u = 0$, we get $A = -\frac{3}{2}$

Putting $u=-\frac{2}{3}$, we get $B=\frac{35}{12}$

and putting $u=\frac{2}{3}$, we get $C=\frac{35}{12}$

$$\therefore \int\frac{4u^2+6}{u(9u^2-4)}du$$

$$=-\frac{3}{2}\int\frac{du}{u}+\frac{35}{12}\int\frac{du}{3u+2}+\frac{35}{12}\int\frac{du}{3u-2}$$

$$=-\frac{3}{2}\log|u|+\frac{35}{36}\log(3u+2)+\frac{35}{36}\log|3u-2|+C$$

$$=-\frac{3}{2}\log e^x+\frac{35}{36}\log(3e^x+2)+\frac{35}{36}\log(3e^x-2)+C$$

$$=-\frac{3}{2}x+\frac{35}{36}\log(9e^{2x}-4)+C$$

$$\therefore A+B=-\frac{3}{2}+\frac{35}{36}=-\frac{19}{36}$$

4. (c) : In $(-1, 0)$, $2-x$ is maximum among $2-x$, 2, $1+x$

In $(0, 1)$, 2 is maximum

$$\therefore \int_{-1}^{1}f(x)dx=\int_{-1}^{0}(2-x)\,dx+\int_{0}^{1}2dx=\frac{9}{2}$$

5. (b) : $I_1=\int_{\sin^2 t}^{1+\cos^2 t}x\,f\{x(2-x)\}dx$

$$=\int_{\sin^2 t}^{1+\cos^2 t}(2-x)\,f\{(2-x)x\}dx$$

$$=2\int_{\sin^2 t}^{1+\cos^2 t}f\{x(2-x)\}dx-I_1$$

$$\therefore 2I_1=2I_2 \Rightarrow \frac{I_1}{I_2}=1$$

6. (d) : The given differential equation is

$$\sqrt{\frac{dy}{dx}}-4\frac{dy}{dx}-7x=0 \Rightarrow 4\frac{dy}{dx}+7x=\sqrt{\frac{dy}{dx}}$$

$$\Rightarrow \left(4\frac{dy}{dx}+7x\right)^2=\frac{dy}{dx}$$

Clearly the order and degree of the above differential equation is 1 and 2 respectively.

7. (b) : The given differential equation is

$$x\log x\frac{dy}{dx}+2y=\log x$$

$$\Rightarrow \frac{dy}{dx}+\frac{2}{x\log x}y=\frac{1}{x}$$

$$\therefore \text{I.F.}=e^{\int\frac{2}{x\log x}dx}=e^{2\int\frac{d(\log x)}{\log x}}$$

$$=e^{2\log|\log x|}=e^{\log(\log x)^2}=(\log x)^2$$

8. (d) : $\int_{-\pi}^{\pi}\frac{2x(1+\sin x)}{1+\cos^2 x}dx$

$$=\int_{-\pi}^{\pi}\frac{2x\,dx}{1+\cos^2 x}+\int_{-\pi}^{\pi}\frac{2x\sin x}{1+\cos^2 x}dx$$

$$=0+2\int_{0}^{\pi}\frac{2x\sin x\,dx}{1+\cos^2 x}$$

$$\left[\because \frac{2x}{1+\cos^2 x}\text{ is an odd function and }\frac{2x\sin x}{1+\cos^2 x}\text{ is an even function}\right]$$

$$\therefore \int_{-\pi}^{\pi}\frac{2x(1+\sin x)}{1+\cos^2 x}dx=4\int_{0}^{\pi}\frac{x\sin x}{1+\cos^2 x}dx$$

Let $I=\int_{0}^{\pi}\frac{x\sin x}{1+\cos^2 x}dx$

$$\therefore I=\int_{0}^{\pi}\frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}dx=\pi\int_{0}^{\pi}\frac{\sin x\,dx}{1+\cos^2 x}-I$$

$$\Rightarrow 2I=\pi\int_{0}^{\pi}\frac{\sin x\,dx}{1+\cos^2 x}$$

$$\Rightarrow I=\frac{\pi}{2}\int_{1}^{-1}\left(-\frac{du}{1+u^2}\right)\text{ where }u=\cos x$$

$$\Rightarrow I=\frac{\pi}{2}[\tan^{-1}u]_{-1}^{1}=\frac{\pi}{2}\left(\frac{\pi}{4}-\left(-\frac{\pi}{4}\right)\right)=\frac{\pi^2}{4}$$

$$\therefore \int_{-\pi}^{\pi}\frac{2x\,(1+\sin x)}{1+\cos^2 x}dx=4I=4\times\frac{\pi^2}{4}=\pi^2$$

9. (c) : The given equation of the curve is

$y = a\sqrt{x} + bx$

$\because$ The above curve passes through the point (1, 2)

$\therefore \quad 2 = a + b$ ...(i)

Also $\int_0^4 (a\sqrt{x} + bx)dx = 8 \Rightarrow a \cdot \frac{2}{3}x^{3/2} + b\frac{x^2}{2}\Big|_0^4 = 8$

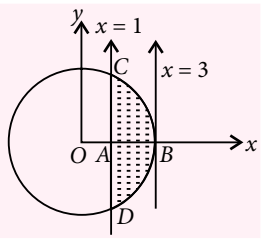
$\Rightarrow \quad a \cdot \frac{2}{3} \cdot 2^3 + 8b = 8$

$\Rightarrow \quad 2a + 3b = 3$ ...(ii)

$\Rightarrow$ Solving (i) and (ii), we get $a = 3$, $b = -1$

10. (b) : The area of the smaller segment cut off from the circle $x^2 + y^2 = 9$ by the line $x = 1$

= area of the shaded region = $2 \times ABCA$



$= 2 \times \int_1^3 \sqrt{9 - x^2}\, dx$ [$\because$ The given circle is $x^2 + y^2 = 9$]

$= 2\left[\frac{x\sqrt{9-x^2}}{2} + \frac{9}{2}\sin^{-1}\frac{x}{3}\right]_1^3$

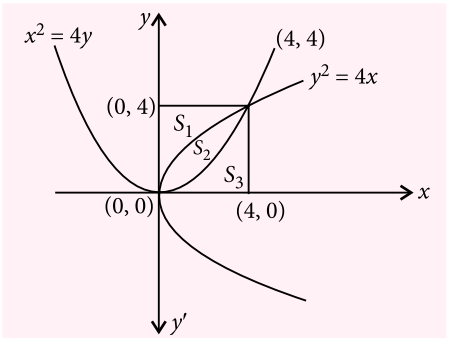
$= 2\left[0 + \left(\frac{9}{2} \times \frac{\pi}{2}\right) - \left(\frac{\sqrt{8}}{2} + \frac{9}{2}\sin^{-1}\frac{1}{3}\right)\right]$

$= \left(9 \times \frac{\pi}{2}\right) - 9\sin^{-1}\frac{1}{3} - \sqrt{8} = 9\left(\frac{\pi}{2} - \sin^{-1}\frac{1}{3}\right) - \sqrt{8}$

$= 9\cos^{-1}\frac{1}{3} - \sqrt{8} = (9\sec^{-1}3 - \sqrt{8})$ sq. units

11. (a, b, c) **12. (a, c)**

13. (a, c, d)



From the diagram, it is clear that

$S_2 = \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4}\right)dx = \left[2 \times \frac{2}{3}x^{3/2} - \frac{1}{4} \cdot \frac{x^3}{3}\right]_0^4$

$= \frac{4}{3}(2^2)^{3/2} - \frac{1}{12} \cdot 4^3 = \frac{32}{3} - \frac{16}{3} \Rightarrow S_2 = \frac{16}{3}$

Again, from the diagram, we find that

$S_1 = S_3 = \frac{1}{2}\left(16 - \frac{16}{3}\right) = \frac{16}{3}$

$\therefore \quad S_1 : S_2 : S_3 = 1 : 1 : 1$

14. (a, b, c) : $\int_0^{\pi/4} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$

$= \int_0^{\pi/4} \frac{\sec^2 x\, dx}{a^2 + b^2\tan^2 x} = \frac{1}{a^2}\int_0^{\pi/4} \frac{\sec^2 x\, dx}{1 + \left(\frac{b}{a}\tan x\right)^2}$

$= \frac{1}{a^2}\frac{a}{b}\int_0^{b/a} \frac{dz}{1+z^2}$ $\left[\begin{array}{l}\text{Put } \frac{b}{a}\tan x = z \\ \frac{b}{a}\sec^2 x dx = dz \\ \sec^2 x dx = \frac{a}{b}dz\end{array}\right]$

$= \frac{1}{ab}\tan^{-1} z\Big|_0^{\frac{b}{a}}$

$= \frac{1}{ab}\tan^{-1}\left(\frac{b}{a}\right)$

15. (a, b, c) : $\int \frac{(3x+4)}{x^3 - 2x - 4}dx$

$= \int \frac{(3x+4)dx}{(x-2)(x^2+2x+2)}$

Let $\frac{3x+4}{(x-2)(x^2+2x+2)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+2x+2}$

$\Rightarrow \quad 3x + 4 = A(x^2 + 2x + 2) + Bx(x - 2) + C(x - 2)$

Solving for A, B and C, we get $A = 1$, $B = C = -1$.

$\therefore \quad \int \frac{3x+4\, dx}{(x-2)(x^2+2x+2)} = \int \frac{dx}{x-2} - \int \frac{x+1}{x^2+2x+2}dx$

$= \log|x-2| - \frac{1}{2}\log|x^2+2x+2| + C$

$\therefore \quad \lambda = -\frac{1}{2}, \; f(x) = x^2 + 2x + 2$

$\therefore \quad f(x) = |x^2 + 2x + 2|$ [$\because x^2 + 2x + 2 > 0 \; \forall x \in R$]

16. (b): $\int\left\{\tan^{-1}\left(\frac{1+\cos x}{\sin x}\right)+\tan^{-1}\left(\frac{\sin x}{1-\cos x}\right)\right\}dx$

$=\int\tan^{-1}\left(\cot\frac{x}{2}\right)dx+\int\tan^{-1}\left(\cot\frac{x}{2}\right)dx$

$=2\int\tan^{-1}\tan\left(\frac{\pi}{2}-\frac{x}{2}\right)dx=2\int\left(\frac{\pi}{2}-\frac{x}{2}\right)dx$

$=2\left[\frac{\pi}{2}x-\frac{1}{4}x^2\right]+c=\pi x-\frac{1}{2}x^2+c$

17. (c)

18. (d): $I=\int_0^{\pi}[\cot x]dx=\int_0^{\pi}[\cot(\pi-x)]dx$

$=\int_0^{\pi}[-\cot x]dx$

$\therefore\quad 2I=\int_0^{\pi}\{[\cot x]+[-\cot x]\}dx$

$2I=\int_0^{\pi}(-1)dx=-\pi$

$[\because\ [x]+[-x]=-1$ for $x\neq$ integer$]$

$\therefore\quad I=-\frac{\pi}{2}$

19. (c): $\int_0^{50}\sin(x-[x])\pi\,dx=50\int_0^{1}\sin\{x\}\,\pi\,dx$

$[\because\ x-[x]=\{x\}$ is a periodic function of period 1$]$

$=50\int_0^{1}\sin\pi x\,dx\quad[\because\{x\}=x$, when $x\in(0,1)]$

$=\frac{50}{\pi}\left[-\cos\pi x\right]_0^1=\frac{50}{\pi}(-\cos\pi+\cos 0)=\frac{100}{\pi}$

20. (b): $dx+xdy=e^{-y}\sec^2 y\,dy$

$\Rightarrow\quad\frac{dx}{dy}+x=e^{-y}\sec^2 y$

Here I.F. $=e^{\int dy}=e^{y}\quad\therefore\quad e^{y}x=\int\sec^2 y\,dy$

$\Rightarrow\quad xe^{y}=\tan y+c$

21. (c): $\cos t\frac{dx}{dt}+x\sin t=1$

$\Rightarrow\quad\frac{dx}{dt}+\tan t\,x=\sec t$

I.F. $=e^{\int\tan t\,dt}=e^{\log\sec t}=\sec t$

$\therefore\quad\frac{d}{dt}(x\sec t)=\sec t.\sec t$

$\Rightarrow\quad x\sec t=\int\sec^2 t\,dt=\tan t+c$

$\Rightarrow\quad x=\sin t+c\cos t$

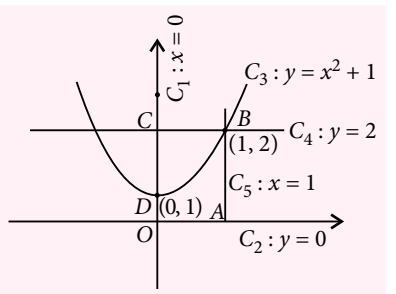
22. (b): The area enclosed by the curves C_1, C_2, C_3 and C_5 is

$OABDO=\int_0^{1}(x^2+1)dx$

$=\left[\frac{1}{3}x^3+x\right]_0^1$

$=\frac{1}{3}+1$

$=\frac{4}{3}$ sq. units



23. (a): Area bounded by the curves C_3 and C_4 is

$2(DBCD)=2\int_1^{2}\sqrt{y-1}\,dy$

$=2\times\frac{2}{3}(y-1)^{3/2}\,|_1^2=\frac{4}{3}$ sq. units

24. A → q; B → r; C → s; D → p

25. A → q; B → r; C → s; D → p

26. A → r; B → s; C → q; D → p

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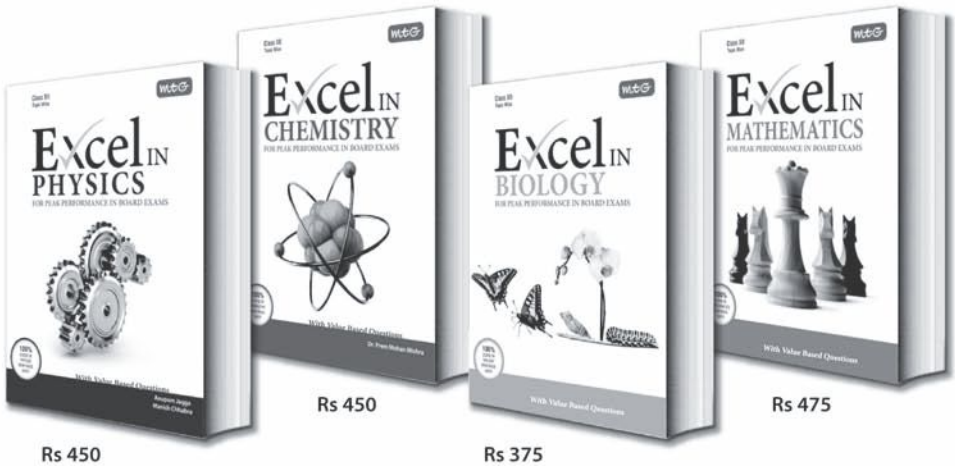

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SAMPLE PAPER

CBSE Board 2015

Time : 3 hrs. Marks : 100

GENERAL INSTRUCTIONS

(i) All questions are compulsory.

(ii) The question paper consists of 29 questions divided into three sections A, B and C. Section A comprises of 10 questions of one mark each, section B comprises of 12 questions of four marks each and section C comprises of 7 questions of six marks each.

(iii) All questions in Section A are to be answered in one word, one sentence or as per the exact requirement of the question.

(iv) There is no overall choice. However, internal choice has been provided in 4 questions of four marks each and 2 questions of six marks each. You have to attempt only one of the alternatives in all such questions.

(v) Use of calculators is not permitted. You may ask for logarithmic tables, if required.

SECTION - A

1. Show that the function $f : R \to R$ defined by $f(x) = \sin x$ is neither one-one nor onto.

2. If the binary operation '$*$' on N (set of natural numbers) be defined as $a * b = 2^{ab}$, then prove that it is commutative but not associative.

3. If A is a matrix of type 3×5 and R is a row of A, then what is the type of R as a matrix?

4. If A, B are skew-symmetric matrices of the same order, prove that AB is symmetric iff A and B commute.

5. Find the derivative of $\sin(\sin x^2)$ at $x = \sqrt{\frac{\pi}{2}}$.

6. Show that $f(x) = x^2$ is strictly decreasing in $(-\infty, 0)$.

7. If $\tan^2\theta = 1 - a^2$, then find the value of $\sec\theta + \tan^3\theta \operatorname{cosec}\theta$.

8. Th e position vectors of three points are $2\vec{a} - \vec{b} + 3\vec{c}$, $\vec{a} - 2\vec{b} + m\vec{c}$ and $n\vec{a} - 5\vec{b}$, where $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors and m, n are scalars. Then prove that the three points are collinear if and only if $m = \frac{9}{4}, n = -2$.

9. If A and B are independent events with $P(A) = 0.6$ and $P(B) = 0.2$, then find $P(A \cup B)$.

10. Find the value of $\lim\limits_{x \to e}\left(\frac{\log x - 1}{x - e}\right)$.

SECTION - B

11. Test the consistency of the following system of equations :

$$3x - y = 5 \text{ and } 6x - 2y = 3$$

12. Evaluate : $\int \frac{\sin(x-a)}{\sin(x+a)}\,dx$.

OR

If $\cos^{-1}\left(\frac{x}{a}\right) + \cos^{-1}\left(\frac{y}{b}\right) = \alpha$, then prove that

$$\frac{x^2}{a^2} - \frac{2xy}{ab}\cos\alpha + \frac{y^2}{b^2} = \sin^2\alpha$$

13. Solve the following differential equation:
$(1 + y^2)(1 + \log x)\,dx + x\,dy = 0$.

14. Find the angle between the line $\vec{r} = (\hat{i} - \hat{j} + \hat{k}) + \lambda(2\hat{i} - \hat{j} + 3\hat{k})$ and the plane $\vec{r} \cdot (2\hat{i} + \hat{j} - \hat{k}) = 4$. Find whether the line is parallel to the plane or not.

15. Find the area cut off from the parabola $4y = 3x^2$ by the line $2y = 3x + 12$.

OR

If a unit vector $\vec{a}$ makes angles $\pi/4$ and $\pi/3$ with x-axis and y-axis respectively and an acute angle θ with z-axis, then find θ and the (scalar and vector) components of $\vec{a}$ along the axes.

16. There is a group of 50 people who are patriotic out of which 20 believe in non-violence. Two persons are selected at random out of them. Write the probability distribution for the selected persons who are non-violent. Also find the mean of the distribution. Explain the importance of non-violence in patriotism.

17. Find the number of roots of the equation $(1 - \cos 2x)\sin 2x = \sqrt{3}\sin^2 x$ which lie in the interval $[-\pi, \pi/3]$.

OR

Prove that :

$$3\tan^{-1}\left(\frac{1}{2}\right)+2\tan^{-1}\left(\frac{1}{5}\right)+\sin^{-1}\left(\frac{142}{65\sqrt{5}}\right)=\pi.$$

18. Prove that the shortest distance of the point $(0, c)$ from the parabola $y = x^2$, where $0 < c \leq 5$ is $\frac{\sqrt{4c-1}}{2}$.

19. Find the value of $\begin{vmatrix} 1 & 2 & 3 \\ 1^3 & 2^3 & 3^3 \\ 1^5 & 2^5 & 3^5 \end{vmatrix}$.

OR

If $f(x)=\begin{vmatrix} (1+x)^{17} & (1+x)^{19} & (1+x)^{23} \\ (1+x)^{23} & (1+x)^{29} & (1+x)^{34} \\ (1+x)^{41} & (1+x)^{43} & (1+x)^{47} \end{vmatrix}$

$= A + Bx + Cx^2 + ...$,

then prove that A is equal to 0.

20. Let $f(x)=\begin{cases} \dfrac{3|x|+4\tan x}{x}, & x \neq 0 \\ k, & x = 0 \end{cases}$, then for what value of k, f is continuous at $x = 0$?

21. Let a relation R on the set A of real numbers be defined as $(a, b) \in R \Rightarrow 1 + ab > 0$ for all $a, b \in A$. Show that R is reflexive and symmetric but not transitive.

22. Using elementary row transformations, find the inverse of matrix $A=\begin{bmatrix} 6 & 5 \\ 5 & 4 \end{bmatrix}$.

SECTION-C

23. A housewife wishes to mix together two kinds of food X and Y in such a way that the mixture contains atleast 10 units of vitamin A, 12 units of vitamin B and 8 units of vitamin C. The vitamin contents of 1 kg of food is given below.

	Vitamin A	Vitamin B	Vitamin C
Food X	1	2	3
Food Y	2	2	1

1 kg of food X cost ₹ 6 and 1 kg of food Y cost ₹ 10. Find the least cost of the mixture which will produce the diet. What is the importance of vitamins in everyday life?

24. Find the vector equation of the line passing through the point (1, 2, –4) and perpendicular to the two lines $\frac{x-8}{3}=\frac{y+19}{-16}=\frac{z-10}{7}$ and $\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}$.

25. Find the local maximum and local minimum values of $\frac{(x-1)(x-6)}{(x-10)}$, $x \neq 10$.

OR

A square piece of tin of side 18 cm is to made into a box without top, by cutting a square from each corner and folding up the flaps to form the box. What should be the side of the square to cut off, so that the volume of the box is the maximum possible? Find the maximum volume.

26. An insurance company insured 2000 cyclists, 4000 scooter drivers and 6000 motobike drivers. The probability of an accident involving a cyclist, scooter driver and a motorbike driver

are 0.01, 0.03 and 0.15 respectively. One of the insured persons meets with an accident. What is the probability that he is a scooter driver? Which mode of transport would you suggest to a student and why?

27. Find the area of the region enclosed between the two circles $x^2 + y^2 = 9$ and $(x - 3)^2 + y^2 = 9$.

OR

Evaluate $\int_1^3 (2x^2 + 5)\,dx$ as limit of sums.

28. Determine whether the following pair of lines intersect or not.

$\vec{r} = \hat{i} - \hat{j} + \lambda(2\hat{i} + \hat{k})$ and $\vec{r} = 2\hat{i} - \hat{j} + \mu(\hat{i} + \hat{j} - \hat{k})$

29. If $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$, then find the product AB and use this result to solve the following system of linear equations $2x - y + z = -1, -x + 2y - z = 4$ and $x - y + 2z = -3$

SOLUTIONS

1. Since $f(0) = \sin 0 = 0$ and $f(\pi) = \sin\pi = 0$, so the different elements 0, π of R have same image. Therefore, f is not one-one.

As $-1 \le \sin x \le 1$ for all $x \in R$, it follows that the range of $f = [-1, 1]$, which is a proper subset of R (co-domain of f). Therefore, f is not onto.

2. For all $a, b \in N$, $a * b = 2^{ab} = 2^{ba} = b * a$

$\Rightarrow$ The given operation is commutative.

As $1, 2, 3 \in N$, $(1 * 2) * 3 = 2^{1\times 2} * 3 = 2^2 * 3$

$= 4 * 3 = 2^{4\times 3} = 2^{12}$

But $1 * (2 * 3) = 1 * 2^{2\times 3} = 1 * 2^6$

$= 1 * 64 = 2^{1\times 64} = 2^{64}$

$\Rightarrow$ The given operation is not associative.

3. Since A is a matrix of type 3×5, therefore, each row of A contains 5 elements. Hence, R is a row matrix of the type 1×5.

4. As A and B are skew-symmetric matrices, therefore, $A' = -A$ and $B' = -B$ ... (i)

Since A and B are square matrices of same order, AB and BA and both are defined.

Now, AB is symmetric iff $(AB)' = AB$

i.e. iff $B'A' = AB$ [reversal law for transposes]

i.e. iff $(-B)(-A) = AB$ [using (i)]

i.e. iff $BA = AB$ *i.e.* iff A and B commute.

5. Let $f(x) = \sin(\sin x^2)$

$\Rightarrow f'(x) = \cos(\sin x^2) \cdot \cos x^2 \cdot 2x$

$\therefore \quad f'\left(\sqrt{\frac{\pi}{2}}\right) = \cos\left(\sin\frac{\pi}{2}\right) \cdot \cos\frac{\pi}{2} \cdot 2 \cdot \sqrt{\frac{\pi}{2}} = 0$

6. Let $x_1, x_2 \in (-\infty, 0)$

Now, consider $x_1 < x_2$ and $x_1 < 0$

$\therefore \quad x_1^2 > x_1 \cdot x_2$ ...(i)

Again consider $x_1 < x_2$ and $x_2 < 0$

$\therefore \quad x_1 \cdot x_2 > x_2^2$ ...(ii)

From (i) and (ii), we get

$x_1^2 > x_2^2 \Rightarrow f(x_1) > f(x_2) \quad [\because f(x) = x^2]$

$\therefore \quad x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

Hence, f is strictly decreasing function in $(-\infty, 0)$

7. $\sec\theta + \tan^3\theta \operatorname{cosec}\theta = \sec\theta(1 + \tan^3\theta \cot\theta)$

$= \sec\theta(1 + \tan^2\theta) = (1 + \tan^2\theta)^{3/2}$

$= (2 - a^2)^{3/2}$

8. Let the points be P, Q and R, respectively. Then P, Q, R are collinear if and only if $\overrightarrow{PQ} = x\overrightarrow{QR}$ for some scalar x. This implies and is implied by

$$-\vec{a} - \vec{b} + (m-3)\vec{c} = x[(n-1)\vec{a} - 3\vec{b} - m\vec{c}]$$

$\Leftrightarrow x(n - 1) = -1, -3x = -1$ and $m - 3 = -xm$

$\Leftrightarrow x = \frac{1}{3}, m = \frac{9}{4}$ and $n = -2$

9. $P(A \cap B) = P(A)P(B) = (0.6)(0.2) = 0.12$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 0.6 + 0.2 - 0.12 = 0.68 = \frac{17}{25}$

10. $\lim_{x\to e} \frac{\log x - 1}{x - e} = \lim_{x\to e} \frac{\log x - \log e}{x - e}$

$= \lim_{h\to 0} \frac{\log(e+h) - \log e}{h} = f'(e)$, where $f(x) = \log x$

$\therefore \quad f'(e) = \frac{1}{e} \qquad \left(\because f'(x) = \frac{1}{x}\right)$

11. Given system of equations are

$3x - y = 5$ ... (i)

and $6x - 2y = 3$ ... (ii)

Given equations can be written in matrix form as $AX = B$

where $A = \begin{bmatrix} 3 & -1 \\ 6 & -2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

Now, $|A| = \begin{vmatrix} 3 & -1 \\ 6 & -2 \end{vmatrix} = -6 + 6 = 0$

$\therefore$ A^{-1} does not exist.

Now, adj$(A) = \begin{bmatrix} -2 & 1 \\ -6 & 3 \end{bmatrix}$

$\therefore$ adj$A(B) = \begin{bmatrix} -2 & 1 \\ -6 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} -10+3 \\ -30+9 \end{bmatrix} = \begin{bmatrix} -7 \\ -21 \end{bmatrix}$

$\therefore$ adj$(A)B \neq O$

Hence, equations are inconsistent and have no solution.

12. Let $I = \int \frac{\sin(x-a)}{\sin(x+a)}\,dx$

Put $x + a = t \Rightarrow x = t - a \Rightarrow dx = dt$

$\therefore \quad I = \int \frac{\sin(t-2a)}{\sin t}\,dt$

$\Rightarrow I = \int \frac{\sin t\cos 2a - \cos t \sin 2a}{\sin t}\,dt$

$\Rightarrow I = \int \frac{\sin t\cos 2a}{\sin t}\,dt - \int \frac{\cos t \sin 2a}{\sin t}\,dt$

$= \cos 2a \int dt - \sin 2a \int \cot t\ dt$

$= t\cos 2a - \sin 2a \log|\sin t| + C$

$= (x + a)\cos 2a - \sin 2a \log|\sin(x + a)| + C$

OR

Given $\cos^{-1}\left(\frac{x}{a}\right) + \cos^{-1}\left(\frac{y}{b}\right) = \alpha$

$\Rightarrow \cos^{-1}\left(\frac{y}{b}\right) = \alpha - \cos^{-1}\left(\frac{x}{a}\right)$

$\Rightarrow \frac{y}{b} = \cos\left(\alpha - \cos^{-1}\left(\frac{x}{a}\right)\right)$

$\Rightarrow \frac{y}{b} = \cos\alpha\left(\frac{x}{a}\right) + \sin\alpha \sin\left(\cos^{-1}\left(\frac{x}{a}\right)\right)$

$\Rightarrow \frac{y}{b} - \frac{x}{a}\cos\alpha = \sin\alpha\sqrt{1-\left(\frac{x}{a}\right)^2}$

$\left(\because \sin(\cos^{-1}x) = \sqrt{1-x^2} \text{ for } |x| \le 1\right)$

$\Rightarrow \left(\frac{y}{b} - \frac{x}{a}\cos\alpha\right)^2 = \sin^2\alpha\left(1 - \frac{x^2}{a^2}\right)$

$\Rightarrow \frac{y^2}{b^2} + \frac{x^2}{a^2}\cos^2\alpha - 2\left(\frac{y}{b}\right)\left(\frac{x}{a}\right)\cos\alpha$

$= \sin^2\alpha\left(1 - \frac{x^2}{a^2}\right)$

$\Rightarrow \frac{x^2}{a^2} - \frac{2xy}{ab}\cos\alpha + \frac{y^2}{b^2} = \sin^2\alpha.$

13. Given differential equation is

$(1 + y^2)\,(1 + \log x)\,dx + x\,dy = 0$

$\Rightarrow x\,dy = -(1 + y^2)\,(1 + \log x)\,dx$

On separating the variables, we get

$\left(\frac{1}{1+y^2}\right)dy = -\frac{(1+\log x)}{x}dx$

$\Rightarrow \left(\frac{1}{1+y^2}\right)dy = -\frac{1}{x}dx - \frac{\log x}{x}dx$

On integrating both sides, we get

$\int \frac{1}{(1+y^2)}dy = -\int \frac{1}{x}dx - \int \frac{\log x}{x}dx$

$\Rightarrow \tan^{-1}y = -\log x - \frac{(\log x)^2}{2} + C$

$\left[\text{Put } \log x = t \Rightarrow \frac{1}{x}dx = dt \right.$

$\left. \therefore \int \frac{\log x}{x}dx = \int t\,dt = \frac{t^2}{2} = \frac{(\log x)^2}{2}\right]$

$\therefore \quad \tan^{-1}y + \log x + \frac{(\log x)^2}{2} = C$

which is the required solution.

14. Equation of line, $\vec{r} = (\hat{i} - \hat{j} + \hat{k}) + \lambda(2\hat{i} - \hat{j} + 3\hat{k})$

Equation of plane, $\vec{r} \cdot (2\hat{i} + \hat{j} - \hat{k}) = 4$.

On comparing with the general equation of a line $\vec{r} = \vec{a} + \lambda\vec{b}$ and the general equation of plane $\vec{r} \cdot \vec{n} = d$, we have $\vec{b} = 2\hat{i} - \hat{j} + 3\hat{k}$ and $\vec{n} = 2\hat{i} + \hat{j} - \hat{k}$.

The angle ϕ between the line and plane is

$$\sin\phi = \left|\frac{\vec{b}\cdot\vec{n}}{|\vec{b}||\vec{n}|}\right|$$

$$= \left|\frac{(2\hat{i} - \hat{j} + 3\hat{k}) \cdot (2\hat{i} + \hat{j} - \hat{k})}{\sqrt{(2)^2 + (-1)^2 + (3)^2}\sqrt{(2)^2 + (1)^2 + (-1)^2}}\right|$$

$$= \left|\frac{(2)(2) + (-1)(1) + (3)(-1)}{\sqrt{4+1+9}\sqrt{4+1+1}}\right| = \left|\frac{4-1-3}{\sqrt{14}\sqrt{6}}\right| = 0$$

$\therefore \quad \phi = 0$

Hence, line is parallel to the plane.

15. Equation of parabola is $y = \frac{3x^2}{4}$ ... (i)

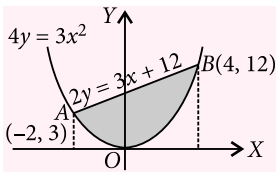
Equation of line is $y = \frac{3x}{2} + 6$ ... (ii)

From (i) and (ii), we get $\frac{3x^2}{4} = \frac{3x}{2} + 6$

$\Rightarrow x^2 - 2x - 8 = 0$

$\Rightarrow (x - 4)(x + 2) = 0 \Rightarrow x = -2, 4$

Here required area $OABO$ is bounded by the curves (i), (ii) and abscissae $x = -2$ and $x = 4$.



Now, the required area $OABO$

$$= \int_{-2}^{4} (y_1 - y_2)\,dx = \int_{-2}^{4} \left[\left(\frac{3x}{2} + 6\right) - \frac{3x^2}{4}\right] dx$$

$$= \left[\frac{3}{4}x^2 + 6x - \frac{x^3}{4}\right]_{-2}^{4} = 27 \text{ sq. units}$$

OR

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, where $\hat{i}$, $\hat{j}$, $\hat{k}$ are unit vectors along the coordinate axes, then

$$\cos\left(\frac{\pi}{4}\right) = \frac{\vec{a}\cdot\hat{i}}{|\vec{a}||\hat{i}|} = a_1$$

$$\therefore \quad a_1 = \frac{1}{\sqrt{2}} \qquad \left[\because |\vec{a}| = 1, |\hat{i}| = 1, \vec{a}\cdot\hat{i} = a_1\right]$$

Similarly, $a_2 = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$ and $a_3 = \cos\theta$

Now, $|\vec{a}| = 1 \Rightarrow \sqrt{a_1^2 + a_2^2 + a_3^2} = 1$

$$\Rightarrow \sqrt{\frac{1}{2} + \frac{1}{4} + \cos^2\theta} = 1 \Rightarrow \frac{3}{4} + \cos^2\theta = 1$$

$$\Rightarrow \cos^2\theta = \frac{1}{4} \Rightarrow \cos\theta = \pm\frac{1}{2}$$

$$\Rightarrow \theta = \frac{\pi}{3}, \frac{2\pi}{3} \qquad [\because 0 \le \theta \le \pi]$$

But θ is acute, $\therefore \theta = \frac{\pi}{3} \therefore a_3 = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

Hence, the scalar components of $\vec{a}$ are $\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2}$ and the vector components are $\frac{1}{\sqrt{2}}\hat{i}, \frac{1}{2}\hat{j}, \frac{1}{2}\hat{k}$.

16. Let X denote the number of non-violent persons.

$$P(X = 0) = \frac{{}^{30}C_2}{{}^{50}C_2} = \frac{30 \times 29}{50 \times 49} = \frac{87}{245}$$

$$P(X = 1) = \frac{{}^{30}C_1 \times {}^{20}C_1}{{}^{50}C_2} = \frac{30 \times 20}{\frac{50 \times 49}{2}} = \frac{120}{245}$$

and $P(X = 2) = \frac{{}^{20}C_2}{{}^{50}C_2} = \frac{20 \times 19}{50 \times 49} = \frac{38}{245}$

Probability distribution of X is as follows:

X	0	1	2
$P(X)$	$\frac{87}{245}$	$\frac{120}{245}$	$\frac{38}{245}$

$$\text{Mean}\left(\bar{X}\right)=\sum P_iX_i=\frac{87}{245}\times 0+\frac{120}{245}\times 1+\frac{38}{245}\times 2=\frac{196}{245}.$$

In order to have a peaceful environment both the values are required patriotism and non-violence. Patriotism alone with violence could be very dangerous.

17. We have,

$$(1-\cos 2x)\sin 2x=\sqrt{3}\sin^2 x=\frac{\sqrt{3}}{2}(1-\cos 2x)$$

$$\therefore\quad (1-\cos 2x)\left(\sin 2x-\frac{\sqrt{3}}{2}\right)=0$$

Two cases arise:

Case 1 : $\cos 2x = 1 \Rightarrow 2x = 2n\pi$ or $x = n\pi$.

This implies $x = -\pi, 0$ when $n = -1, 0$.

Case 2 : $\sin 2x=\frac{\sqrt{3}}{2}$.

$$\Rightarrow\ 2x=k\pi+(-1)^k\left(\frac{\pi}{3}\right);\ k\in Z$$

$$\text{or}\quad x=\frac{k\pi}{2}+(-1)^k\left(\frac{\pi}{6}\right)$$

When $k = -2, -1, 0, 1$ respectively, then

$$x=-\frac{5\pi}{6},\frac{-2\pi}{3},\frac{\pi}{6},\frac{\pi}{3}$$

$$\therefore\quad x=-\pi,\frac{-5\pi}{6},\frac{-2\pi}{3},0,\frac{\pi}{6},\frac{\pi}{3}$$

OR

$$\sin^{-1}\left(\frac{142}{65\sqrt{5}}\right)=\tan^{-1}\left(\frac{142}{\sqrt{(65\sqrt{5})^2-(142)^2}}\right)=\tan^{-1}\left(\frac{142}{31}\right)$$

$$2\tan^{-1}\left(\frac{1}{5}\right)=\tan^{-1}\left(\frac{2/5}{1-(1/25)}\right)=\tan^{-1}\left(\frac{5}{12}\right)$$

$$3\tan^{-1}\left(\frac{1}{2}\right)=\tan^{-1}\left(\frac{(3/2)-(1/8)}{1-(3/4)}\right)=\tan^{-1}\left(\frac{11}{2}\right)$$

Therefore the given sum equals

$$\tan^{-1}\left(\frac{11}{2}\right)+\tan^{-1}\left(\frac{5}{12}\right)+\tan^{-1}\left(\frac{142}{31}\right)+\pi$$

$$=\pi+\tan^{-1}\left(\frac{(11/2)+(5/12)}{1-(55/24)}\right)+\tan^{-1}\left(\frac{142}{31}\right)$$

$$=\pi-\tan^{-1}\left(\frac{142}{31}\right)+\tan^{-1}\left(\frac{142}{31}\right)=\pi$$

18. Let (h, k) be any point on the parabola $y = x^2$. Let D be the required distance between (h, k) and $(0, c)$. Then

$$D=\sqrt{(h-0)^2+(k-c)^2}=\sqrt{h^2+(k-c)^2}\quad \text{... (i)}$$

Since (h, k) lies on the parabola $y = x^2$, we have $k = h^2$.

So (i) gives $D\equiv D(k)=\sqrt{k+(k-c)^2}$

$$D'(k)=\frac{1+2(k-c)}{2\sqrt{k+(k-c)^2}}$$

Now, $D'(k)=0$ gives $k=\frac{2c-1}{2}$

Observe that when $k<\frac{2c-1}{2}$, then

$2(k - c) + 1 < 0$, *i.e.*, $D'(k) < 0$.

Also when $k>\frac{2c-1}{2}$, then $D'(k) > 0$.

So, by first derivative test, $D(k)$ is minimum at $k=\frac{2c-1}{2}$. Hence, the required shortest distance is given by

$$D\left(\frac{2c-1}{2}\right)=\sqrt{\frac{2c-1}{2}+\left(\frac{2c-1}{2}-c\right)^2}=\frac{\sqrt{4c-1}}{2}.$$

19. Let $|A|=\begin{vmatrix}1 & 2 & 3\\ 1^3 & 2^3 & 3^3\\ 1^5 & 2^5 & 3^5\end{vmatrix}=1\cdot 2\cdot 3\begin{vmatrix}1 & 1 & 1\\ 1^2 & 2^2 & 3^2\\ 1^4 & 2^4 & 3^4\end{vmatrix}$

$$=6\begin{vmatrix}1 & 1 & 1\\ 1 & 4 & 9\\ 1 & 16 & 81\end{vmatrix}$$

On applying $C_3 \to C_3 - C_2$, we get

$$|A| = 6\begin{vmatrix} 1 & 1 & 0 \\ 1 & 4 & 5 \\ 1 & 16 & 65 \end{vmatrix}$$

On applying $C_2 \to C_2 - C_1$, we get

$$|A| = 6\begin{vmatrix} 1 & 0 & 0 \\ 1 & 3 & 5 \\ 1 & 15 & 65 \end{vmatrix}$$

$$= 6 \cdot 3 \cdot 5\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 5 & 13 \end{vmatrix} = 90[1(13-5)] = 720 = 6!$$

OR

Given that,

$$f(x) = \begin{vmatrix} (1+x)^{17} & (1+x)^{19} & (1+x)^{23} \\ (1+x)^{23} & (1+x)^{29} & (1+x)^{34} \\ (1+x)^{41} & (1+x)^{43} & (1+x)^{47} \end{vmatrix}$$

$= A + Bx + Cx^2 + ...$

On putting $x = 0$, we have

$$f(0) = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = A$$

$\Rightarrow A = 0$ ($\because R_1$ and R_2 are identical)

20. $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} \dfrac{3|x| + 4\tan x}{x}$

$= \lim\limits_{x \to 0^-} \dfrac{3(-x) + 4\tan x}{x}$ ($\because |x| = -x$ for $x < 0$)

$= \lim\limits_{x \to 0^-} \dfrac{-3x}{x} + \lim\limits_{x \to 0^-} \dfrac{4\tan x}{x} = -3 + 4 = 1$

and $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} \dfrac{3|x| + 4\tan x}{x}$

$= \lim\limits_{x \to 0^+} \left(\dfrac{3x}{x} + \dfrac{4\tan x}{x}\right)$ ($\because |x| = x$ for $x > 0$)

$= 3 + 4 = 7$

$\lim\limits_{x \to 0^-} f(x) \neq \lim\limits_{x \to 0^+} f(x)$

$\therefore$ f is not continuous at 0, whatever k may be.

21. **Reflexive:** Let a be any real number, then

$1 + aa = 1 + a^2 > 0$ ($\because a^2 > 0$ for all $a \in A$)

So, R is reflexive.

Symmetric: Let $(a, b) \in R$, then

$(a, b) \in R \Rightarrow (1 + ab) > 0 = 1 + ba > 0$

($\because ab = ba$ for all $a, b \in A$)

$\Rightarrow (b, a) \in R$

Thus, $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in A$.

Hence, R is symmetric.

Transitive: We observe that

$\left(1, \dfrac{1}{2}\right) \in R$ and $\left(\dfrac{1}{2}, -1\right) \in R$ but $(1, -1) \notin R$

because $1 + 1 \times (-1) = 0 \not> 0$

Hence, R is not transitive on A.

22. We have, $A = \begin{bmatrix} 6 & 5 \\ 5 & 4 \end{bmatrix}$.

Let $A = IA \Rightarrow \begin{bmatrix} 6 & 5 \\ 5 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$

On applying $R_1 \to R_1 - R_2$, we get

$$\begin{bmatrix} 1 & 1 \\ 5 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} A$$

On applying $R_2 \to R_2 - 5R_1$, we get

$$\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -5 & 6 \end{bmatrix} A$$

On applying $R_1 \to R_1 + R_2$, we get

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -4 & 5 \\ -5 & 6 \end{bmatrix} A$$

On applying $R_2 \to (-1)R_2$, we get

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -4 & 5 \\ 5 & -6 \end{bmatrix} A$$

Hence, $A^{-1} = \begin{bmatrix} -4 & 5 \\ 5 & -6 \end{bmatrix}$ ($\because A^{-1}A = I$)

23. Let the quantity of food X be x kg and the quantity of food Y be y kg.

$\therefore$ The objective function is to minimise

$Z = 6x + 10y$.

Subject to the constraints

$x + 2y \geq 10, 2x + 2y \geq 12$

$3x + y \geq 8, x, y \geq 0$

Consider the constraints as equations, we get

$x + 2y = 10$ ... (i)

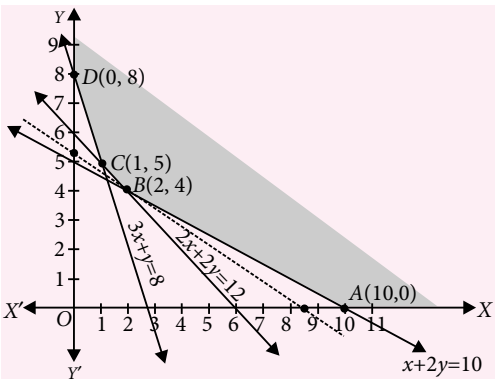
$2x + 2y = 12$ or $x + y = 6$ ... (ii)

$3x + y = 8$ ... (iii)

and $x = 0, y = 0$ ... (iv)

Let us draw the graph of (i), (ii), (iii) and (iv).

The graphical representation of these lines is given below.



The shaded region in the graph represents the feasible region and its extreme points are $A(10, 0)$, $B(2, 4)$, $C(1, 5)$ and $D(0, 8)$.

Now the values of $Z = 6x + 10y$ at extreme points are

At $A(10, 0)$; $Z = 6(10) + 10(0) = 60$

At $B(2, 4)$; $Z = 6(2) + 10(4) = 52$

At $C(1, 5)$; $Z = 6(1) + 10(5) = 56$

At $D(0, 8)$; $Z = 6(0) + 10(8) = 80$

$\therefore$ Z is minimum at $x = 2$ and $y = 4$ and minimum value of Z is ₹ 52.

As the feasible region is unbounded, therefore 52 may or may not be the minimum value of Z. For this we draw a dotted graph of the inequality, $6x + 10y < 52$ or $3x + 5y < 26$ and check, whether the resulting half plane has points in common with the feasible region or not. It can be seen that the feasible region has no common point with $3x + 5y < 26$. Therefore, the minimum value of Z is 52 at $B(2, 4)$.

Hence, the mixture should contain 2 kg of food X and 4 kg of food Y. The minimum cost of the mixture is ₹ 52.

Vitamins prevents us from falling sick and keep us healthy. For example, vitamin A is good for eyes, vitamin C is good for bones, teeth and gums.

24. Equation of line passing through (1, 2, – 4) is

$$\frac{x-1}{a} = \frac{y-2}{b} = \frac{z+4}{c} \quad \text{... (i)}$$

Since, line (i) is perpendicular to the following given lines.

$$\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7} \quad \text{... (ii)}$$

and $\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$ ... (iii)

$\therefore$ $3a - 16b + 7c = 0$ ... (iv)

and $3a + 8b - 5c = 0$ ... (v)

On solving (iv) and (v), we get

$$\frac{a}{80-56} = \frac{-b}{-15-21} = \frac{c}{24+48}$$

$$\Rightarrow \frac{a}{24} = \frac{-b}{-36} = \frac{c}{72} \Rightarrow \frac{a}{2} = \frac{b}{3} = \frac{c}{6} = \lambda$$

$\therefore$ $a = 2\lambda$, $b = 3\lambda$ and $c = 6\lambda$

On putting these values of a, b and c in (i), we get

$$\frac{x-1}{2\lambda} = \frac{y-2}{3\lambda} = \frac{z+4}{6\lambda}$$

$$\therefore \quad \frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6}$$

which is the required cartesian equation of line.

Now, $\vec{a} = \hat{i} + 2\hat{j} - 4\hat{k}$

(coordinate of a point in vector form)

and $\vec{b} = 2\hat{i} + 3\hat{j} + 6\hat{k}$

(DR's of the line in vector form)

So, vector equation of line is

$$\vec{r} = \vec{a} + \lambda\vec{b}$$

$$\therefore \quad \vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$$

25. Let $y = \frac{(x-1)(x-6)}{x-10} = \frac{x^2 - 7x + 6}{x-10}$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{(x-10)(2x-7) - (x^2 - 7x + 6)\cdot 1}{(x-10)^2}$$

$$= \frac{x^2 - 20x + 64}{(x-10)^2} = \frac{(x-4)(x-16)}{(x-10)^2}$$

For stationary points,

$$\frac{dy}{dx}=0 \quad \Rightarrow \quad \frac{(x-4)(x-16)}{(x-10)^2}=0$$

$\Rightarrow (x-4)(x-16)=0$

$\therefore \; x=4, 16$

(i) For $x=4$

When x is slightly < 4, then

$$\frac{dy}{dx}=\frac{(-)(-)}{(+)} = \text{Positive}$$

When x is slightly > 4, then

$$\frac{dy}{dx}=\frac{(+)(-)}{(+)} = \text{Negative}$$

$\therefore \; \frac{dy}{dx}$ changes sign from positive to negative.

$\therefore \; y$ is locally maximum at $x=4$.

Local maximum value at $x=4$ is

$$\frac{(4-1)(4-6)}{4-10}=\frac{3\times(-2)}{-6}=1$$

(ii) For $x=16$

When x is slightly < 16, then

$$\frac{dy}{dx}=\frac{(+)(-)}{(+)} = \text{Negative}$$

When x is slightly > 16, then

$$\frac{dy}{dx}=\frac{(+)(+)}{(+)} = \text{Positive}$$

$\therefore \; \frac{dy}{dx}$ changes sign from negative to positive.

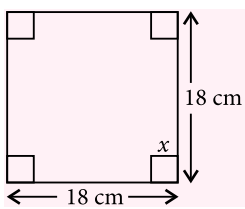
Hence y is locally minimum at $x=16$.

Local minimum value at $x=16$ is

$$\frac{(16-1)(16-6)}{16-10}=\frac{15\times10}{6}=25$$

OR

Let the side to be cut be x cm.



$\therefore$ Length of box, $l = 18-2x$

Breadth of box, $b = 18-2x$

Height of box, $h = x$

So, volume of box, $V = lbh$

$V=(18-2x)\cdot(18-2x)\cdot x$

$=4(9-x)^2\cdot x$ ... (i)

$=4(81-18x+x^2)\cdot x$

$V=4(x^3-18x^2+81x)$

On differentiating V w.r.t x, we get

$$\frac{dV}{dx}=4(3x^2-36x+81) = 4\cdot3(x^2-12x+27)$$

$$=12(x^2-12x+27) \quad \text{... (ii)}$$

For maximum or minimum volume V, we have

$$\frac{dV}{dx}=0 \Rightarrow x^2-12x+27=0$$

$\Rightarrow \quad (x-9)(x-3)=0 \Rightarrow x=9, 3$

So, the stationary points are $x=9$ and $x=3$.

But $x=9$ is not possible.

Thus, $x=3$

Now, again differentiating (ii), we get

$$\frac{d^2V}{dx^2}=12(2x-12)=24(x-6)$$

$$\Rightarrow \left[\frac{d^2V}{dx^2}\right]_{x=3} = 24(3-6)=24(-3)=-72<0$$

$\therefore$ Volume is maximum at $x=3$ cm.

Maximum volume $= 4(9-3)^2\cdot3 = 4\cdot36\cdot3$

$= 432 \text{ cm}^3$ [using eq. (i)]

26. Let E_1 : 'Insured person is cyclist',

E_2 : 'Insured person is a scooter driver'

and E_3 : 'Insured person is a motorbike driver',

then E_1, E_2, E_3 are mutually exclusive and exhaustive.

Also, $P(E_1)=\frac{2000}{12000}=\frac{1}{6}$, $P(E_2)=\frac{4000}{12000}=\frac{1}{3}$

and $P(E_3)=\frac{6000}{12000}=\frac{1}{2}$.

Let E : 'An insured driver meets with an accident',

then

$P(E/E_1)=0.01$, $P(E/E_2)=0.03$ and $P(E/E_3)=0.15$.

Using Bayes' Theorem,

$$P(E_2/E)=\frac{P(E/E_2)\,P(E_2)}{\{P(E/E_1)\,P(E_1)+P(E/E_2)\,P(E_2)+P(E/E_3)\,P(E_3)\}}$$

$$=\frac{0.03\times\frac{1}{3}}{0.01\times\frac{1}{6}+0.03\times\frac{1}{3}+0.15\times\frac{1}{2}}$$

$$=\frac{6}{1+6+45}=\frac{6}{52}=\frac{3}{26}$$

Similarly, $P(E_1/E)=\frac{1}{52}$ and $(E_3/E)=\frac{45}{52}$

We advise the students to use bicycle only as a means of conveyance as it is environment friendly, cheaper and keeps the body fit. Also the probability of a cyclist being involved in an accident is least equal to $\frac{1}{52}$.

27. The given circles are

$x^2+y^2=9$ ...(i)

and $(x-3)^2+y^2=9$ ...(ii)

(i) represents a circle with centre (0, 0) and radius 3 units and (ii) represents a circle with centre (3, 0) and radius 3 units.

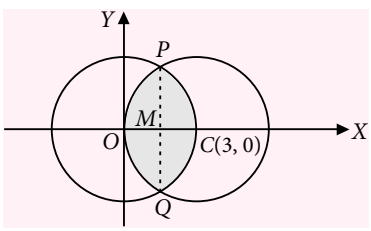
On solving (i) and (ii), we get,

$$x=\frac{3}{2},\ y=\pm\frac{3\sqrt{3}}{2}$$

$\therefore$ The points of intersection of (i) and (ii) are

$$P\left(\frac{3}{2},\frac{3\sqrt{3}}{2}\right),\ Q\left(\frac{3}{2},\frac{3\sqrt{3}}{2}\right).$$

The area of the region enclosed between the given circles is shown shaded in the figure.



From, (i) $y=\sqrt{9-x^2}$ (in the first quadrant)

$\therefore$ Required area = 2(area of the region *OMCPO*)

= 2 × 2 (area of region *MCPM*)

$$=4\int_{3/2}^{3}\sqrt{9-x^2}\,dx$$

$$=4\left[\frac{x\sqrt{9-x^2}}{2}+\frac{3^2}{2}\sin^{-1}\left(\frac{x}{3}\right)\right]_{3/2}^{3}$$

$$=2\left((0+9\sin^{-1}(1))-\left(\frac{3}{2}\sqrt{9-\frac{9}{4}}+9\sin^{-1}\left(\frac{1}{2}\right)\right)\right)$$

$$=2\left(9\times\frac{\pi}{2}-\frac{9\sqrt{3}}{4}-9\times\frac{\pi}{6}\right)=6\pi-\frac{9\sqrt{3}}{2}$$

Hence, the required area

$$=\left(6\pi-\frac{9}{2}\sqrt{3}\right)\text{sq. units.}$$

OR

We have

$$\int_a^b f(x)dx=\lim_{h\to0}h[f(a)+f(a+b)+....+f(a+(n-1)h)]$$

where $h=\frac{b-a}{n}$

$a=1, b=3, f(x)=2x^2+5$ and $h=\frac{3-1}{n}=\frac{2}{n}$

$$\therefore\ I=\lim_{h\to0}[f(1)+f(1+h)+...+f(1+(n-1)h)]$$

$$\Rightarrow\ I=\lim_{h\to0}h[(2(1)^2+5)+(2(1+h)^2+5)+...+(2(1+(n-1)h^2)+5)]$$

$$\Rightarrow\ I=\lim_{h\to0}h[2(1^2+(1+h)^2+...+(1+(n-1)h)^2+5n]$$

$$\Rightarrow\ I=\lim_{h\to0}h[2(n+2h(1+2+...+(n-1))+h^2(1^2+2^2+...+(n-1)^2)+5n]$$

$$=\lim_{h\to0}h\left[2\left(n+2h\cdot\frac{n(n-1)}{2}+h^2\,\frac{n(n-1)(2n-1)}{6}\right)+5n\right]$$

$$= \lim_{h\to 0} h\left[2n + 2nh(n-1) + \frac{2h^2 n(n-1)(2n-1)}{6} + 5n\right]$$

$$= \lim_{n\to\infty} \frac{2}{n}\left[7n + 2\times\frac{2}{n}\times n(n-1) + 2\left(\frac{4}{n^2}\right)\frac{n(n-1)(2n-1)}{6}\right]$$

$$= \lim_{n\to\infty}\left[14 + 8\left(\frac{n-1}{n}\right) + \frac{8}{3}\frac{(n-1)(2n-1)}{n^2}\right]$$

$$\Rightarrow \quad I = \lim_{n\to\infty}\left[14 + 8\left(1-\frac{1}{n}\right) + \frac{8}{3}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)\right]$$

$$\Rightarrow \quad I = 14 + 8(1-0) + \frac{8}{3}(1-0)(2-0)$$

$$= 14 + 8 + \frac{16}{3} = \frac{82}{3}$$

28. The shortest distance between the given pair of lines is given by

$$d = \left|\frac{(\vec{a}_2 - \vec{a}_1)\cdot(\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|}\right|$$

The two lines will intersect if and only if $d = 0$,

i.e., $(\vec{a}_2 - \vec{a}_1)\cdot(\vec{b}_1 \times \vec{b}_2) = 0$

Here equation of first line is

$\vec{r} = \hat{i} - \hat{j} + \lambda(2\hat{i} + \hat{k}) = \vec{a}_1 + \lambda\vec{b}_1$

where $\vec{a}_1 = \hat{i} - \hat{j}$ and $\vec{b}_1 = 2\hat{i} + \hat{k}$

Also equation of second line is

$\vec{r} = 2\hat{i} - \hat{j} + \mu(\hat{i} + \hat{j} - \hat{k}) = \vec{a}_2 + \mu\vec{b}_2$

where $\vec{a}_2 = 2\hat{i} - \hat{j}$ and $\vec{b}_2 = \hat{i} + \hat{j} - \hat{k}$

$$\text{Now, } \vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & 1 \\ 1 & 1 & -1 \end{vmatrix} = -\hat{i} + 3\hat{j} + 2\hat{k}$$

and $\vec{a}_2 - \vec{a}_1 = 2\hat{i} - \hat{j} - (\hat{i} - \hat{j}) = \hat{i}$

$(\vec{a}_2 - \vec{a}_1)\cdot(\vec{b}_1 \times \vec{b}_2) = \hat{i}\cdot(-\hat{i} + 3\hat{j} + 2\hat{k})$

Hence the given lines do not intersect.

29. Given,

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$$\text{Now, } AB = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 6-1-1 & 2-3+1 & -2-1+3 \\ -3+2+1 & -1+6-1 & 1+2-3 \\ 3-1-2 & 1-3+2 & -1-1+6 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = 4I$$

$\Rightarrow AB = 4I$

On multiplying both sides by A^{-1}, we get

$A^{-1}(AB) = A^{-1}(4I) \Rightarrow (A^{-1}A)B = 4A^{-1}I$

$$\Rightarrow IB = 4A^{-1} \Rightarrow A^{-1} = \frac{1}{4}B$$

The given system of equations are

$2x - y + z = -1$, $-x + 2y - z = 4$ and $x - y + 2z = -3$

In matrix form, it can be written as

$A \cdot X = C \Rightarrow X = A^{-1} \cdot C$

where, $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and

$C = \begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -3+4+3 \\ -1+12-3 \\ 1+4-9 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

On comparing corresponding elements, we get

$x = 1$, $y = 2$ and $z = -1$

MATHS MUSING

SOLUTION SET-145

1. (a): $\frac{x+y}{2}-\sqrt{xy}=2$

$\therefore \quad (\sqrt{y}-\sqrt{x})^2=4, \sqrt{y}=2+\sqrt{x}$

There are 42 pairs : $(1^2, 3^2)$, $(2^2, 4^2)$, $(3^2, 5^2)$, ..., $(42^2, 44^2)$

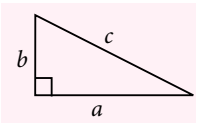
2. (b) : $xy = 2015(x + y)$

$\therefore \quad (x - 2015)(y - 2015) = 2015^2 = 5^2 \cdot 13^2 \cdot 31^2$.

With number of divisors $= 3 \times 3 \times 3 = 27$

$\therefore$ Number of pairs (x, y) is $\frac{2T-1}{2}=13$.

3. (d) : $r=\frac{\Delta}{s}=\frac{ab}{a+b+c}$



$\Rightarrow \quad (a - 2r)(b - 2r) = 2r^2 = 2 \cdot 5^2 \cdot 13^2 \cdot 31^2$.

The number of triangles is $\frac{1}{2}$ the number of divisors

$=\frac{1}{2}\times 2\times 3\times 3\times 3 = 27$

4. (a): $16x + 25y + 36z = 305 + 3(880 - 310) = 2015$

5. (d) : If α, β are the roots, then $\alpha + \beta = -a$, $\alpha\beta = 6a \Rightarrow (\alpha + 6)(\beta + 6) = 36$ gives integer roots for 10 values of a : $-25, -8, -3, -1, 0, 24, 25, 27, 32, 49$.

6. (a, b, c, d) :

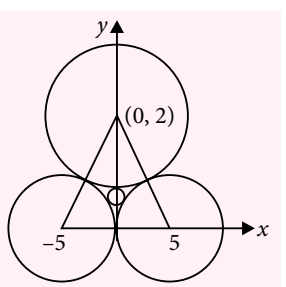
$f'(x) > 0$ in $[-1, 2]$. $f(x)$ increases

$f'(2^-) = f(2) = f(2^+)$. $f(x)$ is continuous

$f'(2^-) = 24$, $f'(2^+) = -1$. $f'(2)$ does not exist.

$f(2) = 35$ is the maximum value.

7. (c): Let $(0, 2)$ be the centre and r, radius of the circle $25 + \alpha^2 = (r + 5)^2 \Rightarrow \alpha^2 = r^2 + 10r$



$12 - \alpha = r + 8 \Rightarrow \alpha = 4 - r$

$(4 - r)^2 = r^2 + 10r \Rightarrow r = \frac{8}{9}$

8. (b) : Let $(0, \beta)$ be the centre and R be the radius of the desired circle

$\therefore \quad 25 + \beta^2 = (R - 5)^2 \Rightarrow \beta^2 = R^2 - 10R$

$12 - \beta = R - 8 \Rightarrow \beta = 20 - R$

$\therefore \quad (20 - R)^2 = R^2 - 10R \Rightarrow R = \frac{40}{3}$

9. (9): The possible number of heads are 0, 1, 2, 3, 4, 5 and rest are tails. Head occur between tails.

$\therefore \quad p=\frac{1}{2^{10}}\sum_{r=0}^{5}\binom{10-r+1}{r}$

$=\frac{1}{2^{10}}\left[\binom{11}{0}+\binom{10}{1}+\binom{9}{2}+\binom{8}{3}+\binom{7}{4}+\binom{6}{5}\right]$

$=\frac{1}{2^{10}}(1+10+36+56+35+6)=\frac{9}{64}\Rightarrow m=9$

10. (d) :

P. $\int_{-1}^{1}\cos^{-1}x\,dx=\cos^{-1}x\cdot x\Big|_{-1}^{1}+\int_{-1}^{1}\frac{x}{\sqrt{1-x^2}}dx=\pi+0=\pi$

Q. $\int_{-1}^{1}\sin^{-1}x\,dx=0$. Since $\sin^{-1}x$ is odd.

R. $\int_{-1}^{1}x\cos^{-1}x\,dx=\frac{x^2}{2}\cos^{-1}x\Big|_{-1}^{1}+\int_{-1}^{1}\frac{x^2}{2\sqrt{1-x^2}}dx$

$=-\frac{\pi}{2}+\frac{\pi}{4}=-\frac{\pi}{4}$

S. $\int_{-1}^{1}x\sin^{-1}x\,dx=\frac{x^2}{2}\sin^{-1}x\Big|_{-1}^{1}-\int_{-1}^{1}\frac{x^2}{2\sqrt{1-x^2}}dx$

$=\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}$

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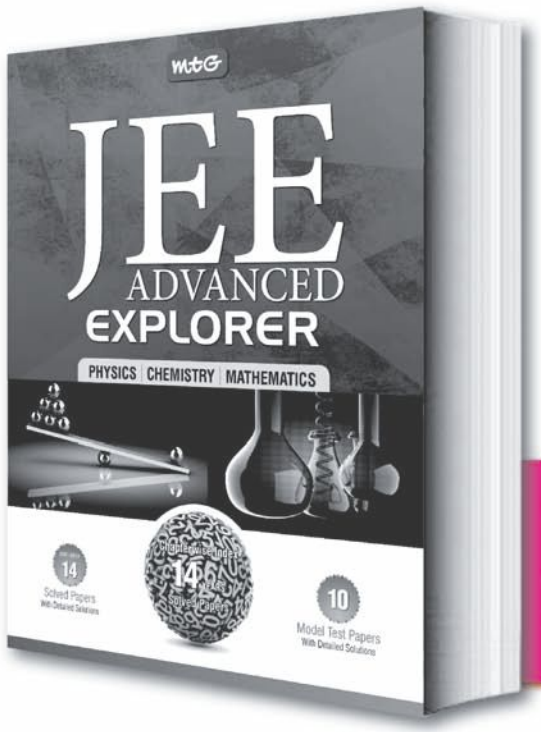

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Q1. Show that the difference of the tangents of the angles which the lines $x^2(\sec^2\theta - \sin^2\theta) - 2xy\tan\theta + y^2\sin^2\theta = 0$ make with x-axis is 2.

- Reema Gupta, Lucknow

Ans. We have $x^2(\sec^2\theta - \sin^2\theta) - 2xy\tan\theta + y^2\sin^2\theta = 0$. Comparing this equation with $ax^2 + 2hxy + by^2 = 0$, we have

$a = \sec^2\theta - \sin^2\theta$, $2h = -2\tan\theta$ and $b = \sin^2\theta$

Let m_1 and m_2 be the slopes of the lines represented by the given equation. Then,

$$m_1 + m_2 = -\frac{2h}{b} = \frac{2\tan\theta}{\sin^2\theta} = \frac{2}{\sin\theta\cos\theta}$$

and, $$m_1 m_2 = \frac{a}{b} = \frac{\sec^2\theta - \sin^2\theta}{\sin^2\theta}$$

$$= \frac{1}{\sin^2\theta - \cos^2\theta} - 1.$$

$$\therefore \quad m_1 m_2 = \sqrt{(m_1 + m_2)^2 - 4m_1 m_2}$$

$$= \sqrt{\frac{4}{\sin^2\theta\cos^2\theta} - \frac{4}{\sin^2\theta\cos^2\theta} + 4} = 2$$

Q2. Find the least value of the function

$$\phi(x) = \int_{5\pi/4}^{x} (3\sin t + 4\cos t)dt$$

on the interval $[5\pi/4, 4\pi/3]$.

- Sachin Khanna, Muzaffarnagar

Ans. $$\phi(x) = \int_{5\pi/4}^{x} (3\sin t + 4\cos t)dt$$

$\Rightarrow \quad \phi'(x) = (3\sin x + 4\cos x)$

Clearly, $\phi'(x) < 0$ for all $x \in [5\pi/4, 4\pi/3]$

$\Rightarrow$ $\phi(x)$ is a decreasing function on $[5\pi/4, 4\pi/3]$.

$\Rightarrow$ $\phi(4\pi/3)$ is the last value of $\phi(x)$ on $[5\pi/4, 4\pi/3]$

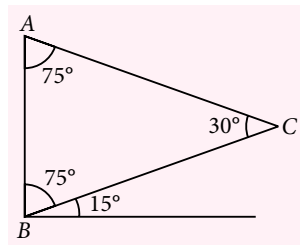
Now, $$\phi(4\pi/3) = \int_{5\pi/4}^{4\pi/3} (3\sin t + 4\cos t)dt$$

$$= [-3\cos t + 4\sin t]_{5\pi/4}^{4\pi/3} = \frac{3}{2} - 2\sqrt{3} + \frac{1}{\sqrt{2}}.$$

Q3. A vertical tower stands on a declivity which is inclined at 15° to the horizon. From the foot of the tower a man ascends the declivity for 80 feet and then finds that the tower subtends an angle of 30°. Find the height of the tower.

- Sneha Chatterjee, Kolkata

Ans. Let BC be the declivity and BA be the tower. Using sine rule in ΔABC, we have



$$\frac{BC}{\sin 75^\circ} = \frac{AB}{\sin 30^\circ} \Rightarrow AB = \frac{80\sin 30^\circ}{\sin 75^\circ}$$

$$\Rightarrow AB = \frac{40\times 2\sqrt{2}}{\sqrt{3}+1} = 40(\sqrt{6} - \sqrt{2}).$$

VIT University Chancellor Dr. G. Viswanathan met the Hon'ble Minister for Human Resource and Development, Mrs. Smriti Zubin Irani in New Delhi recently and honoured her on assuming as Union Minister.

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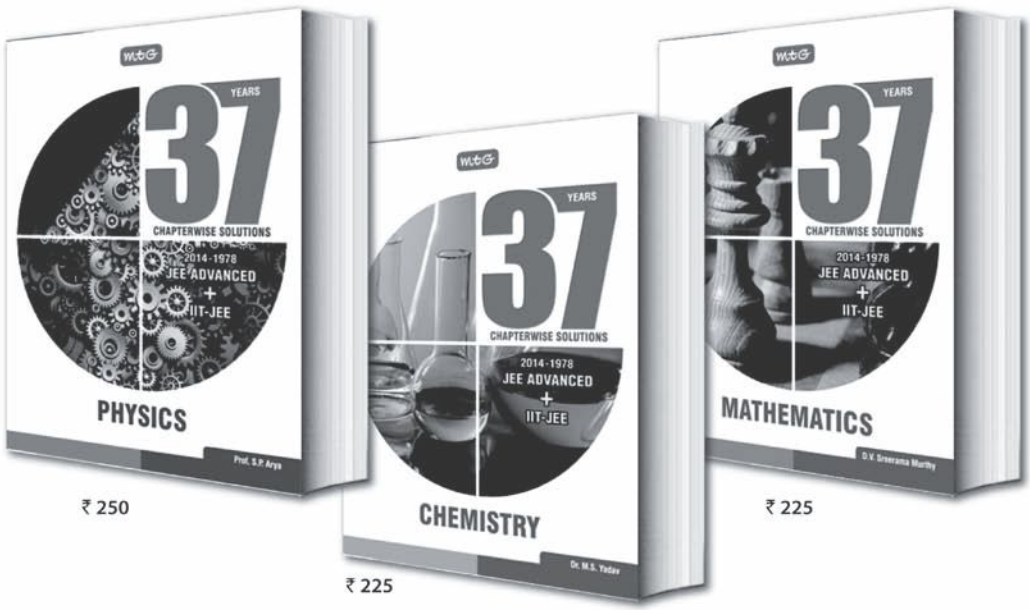

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OLYMPIAD CORNER

1. In Lotto $\frac{6}{49}$ six balls are randomly drawn (without replacement) from a bin holding balls numbered from 1 to 49. Find the expected value of the k^{th} lowest number draw, for each $k = 1, 2, ..., 6$.

2. Let n be a positive integer and let $W = ...x_{-1}x_0x_1x_2 ...$ be an infinite periodic word consisting of the letters a and b. Suppose that the minimal period N of W is greater than 2^n.
   A finite non-empty word U is said to appear in W if there exist indices $k \le l$ such that $U = x_kx_{k+1} ...x_l$. A finite word U is called ubiquitous if the four words Ua, Ub, aU and bU all appear in W. Prove that there are atleast n ubiquitous finite non-empty words.

3. Prove or disprove that :
   $$\sqrt{5+\sqrt{21}}+\sqrt{8+\sqrt{55}}=\sqrt{7+\sqrt{33}}+\sqrt{6+\sqrt{35}}$$

4. A convex quadrilateral $ABCD$ with $AC \neq BD$ is inscribed in a circle with centre O, and E is the intersection of diagonals AC and BD. Let P be an interior point of $ABCD$ such that $\angle PAB + \angle PCB = \angle PBC + \angle PDC = 90°$
   Prove that O, P and E are collinear.

5. The sum of two consecutive squares can be a square ($3^2 + 4^2 = 5^2$).
   (i) Prove that the sum of m consecutive squares cannot be a square for the cases $m = 3, 4, 5, 6$.
   (ii) Find an example of eleven consecutive squares whose sum is a square.

SOLUTIONS

1. We consider more generally "Lotto n/N" (where n balls are drawn from balls numbered from 1 to N, $n \le N$), and show that the expected value of the k^{th} lowest number drawn equals
   $$E(k)=\frac{k(N+1)}{n+1},\ k=1,2,....,n$$
   Indeed, x is the k^{th} lowest from $\binom{x-1}{k-1}\binom{N-x}{n-k}$ draws (since for each of the $k-1$ numbers drawn from the $x-1$ numbers less than x there are $n-k$ numbers drawn from the $N-x$ numbers greater than x). Thus, the probability that x is the k^{th} lowest number equals
   $$\frac{\binom{x-1}{k-1}\binom{N-x}{n-k}}{\binom{N}{n}},\ k \le x \le N-n+k$$
   So that
   $$\sum_{x=k}^{N-n+k}\binom{x-1}{n-k}\binom{N-x}{n-k}=\binom{N}{n}$$
   Note that this argument provides a neat proof of the familiar identity
   $$\sum_{x=b}^{a-c}\binom{x}{b}\binom{a-x}{c}=\binom{a-1}{b+c+1}$$
   It follows that $E(k)=\sum_{x=k}^{N-n+k}\frac{\binom{x-1}{k-1}\binom{N-x}{n-k}}{\binom{N}{n}}$

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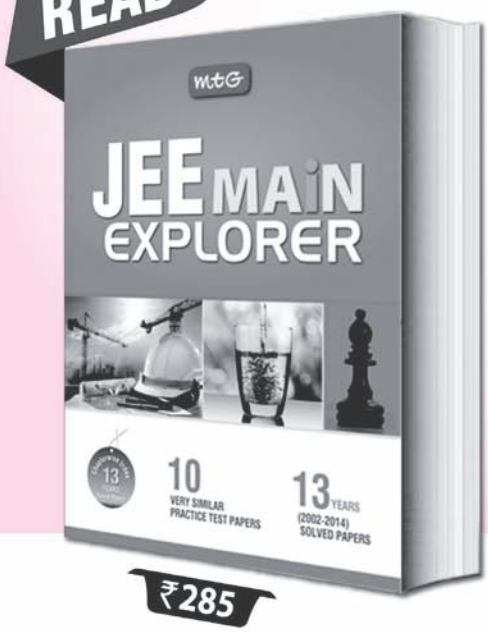

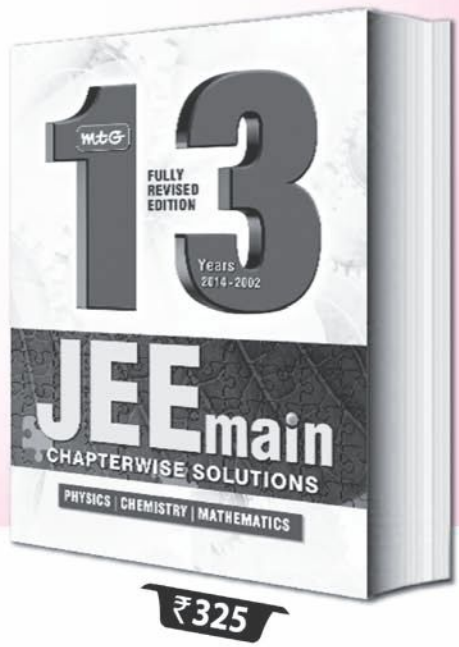

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$$= k\sum_x \frac{\binom{x}{k}\binom{N-x}{n-k}}{\binom{N}{n}} = \frac{k\binom{N+1}{n+1}}{\binom{N}{n}} = \frac{k(N+1)}{n+1}$$

as claimed.

In Lotto $\frac{6}{49}$ the expected values are therefore

$$E(k) = \frac{50k}{7} = \frac{50}{7}, \frac{100}{7}, ..., \frac{300}{7}$$

2. Throughout the solution, all the words are non-empty. For any word R of length m, we call the number of indices $i \in \{1, 2, ..., N\}$ for which R coincides with the subword $x_{i+1}x_{i+2} \dots \dots x_{i+m}$ of W the multiplicity of R and denote it by $\mu(R)$. Thus a word R appears in W if and only if $\mu(R) > 0$. Since each occurrence of a word in W is both succeeded by either the letter a or the letter b and similarly preceded by one of those two letters, we have

$\mu(R) = \mu(Ra)\ \mu(Rb) = \mu(aR) + \mu(bR)$ ...(i)

for all words R.

We claim that the condition that N is in fact the minimal period of W guarantee that each word of length N has multiplicity 1 or 0 depending on whether it appears or not. Indeed, if the words $x_{i+1}x_{i+2} \dots x_{i+N}$ and $x_{j+1} \dots x_{j+N}$ are equal for some $1 \le i < j \le N$, then we have $x_{i+a} = x_{j+a}$ for every integer a, and hence $j - i$ is also a period.

Moreover, since $N > 2^n$, at least one of the two words a and b has a multiplicity that is strictly larger than 2^{n-1}.

For each $k = 0, 1, ..., n - 1$, let U_k be a subword of W whose multiplicity is strictly larger than 2^k and whose length is maximal subject to this property. Note that such a word exists in view of the two observations made in the two previous paragraphs.

Fix some index $k \le \{0, 1, n - 1\}$. Since the word U_kb is longer than U_k, its multiplicity can be at most 2^k, so in particular $\mu(U_kb) < \mu(U_k)$. Therefore, the word U_ka has to appear by (1). For a similar reason, the words U_kb, aU_k, and bU_k have a appear as well. Hence, the word U_k is ubiquitous. Moreover, if the multiplicity of U_k were strictly greater than 2^{k+1}, then by (1) at least one of the two words U_ka and U_kb would have multiplicity greater than 2^k and would thus violate the maximality condition imposed on U_k.

So we have $\mu(U_0) \le 2 \le \mu(U_1) \le 4 < .... \le 2^{n-1} < \mu(U_{n-1})$, which implies in particular that the words U_0, U_1, ..., U_{n-1} have to be distinct. As they have been proved to be ubiquitous as well, the problem is solved.

3. In order to simplify the radicals, the radicands should be forced to equal square numbers (*e.g.*, $7 + \sqrt{33}$ should be a square of some number). Numbers whose squares have a rational and radical part are usually in the form $a + b$.

So let, $\sqrt{7+\sqrt{33}} = a + b = \sqrt{(a+b)^2}$

$$= \sqrt{a^2 + b^2 + 2ab},$$

and set $a^2 + b^2 = 7$ and $2ab = \sqrt{33}$

i.e. $b = \frac{\sqrt{33}}{2a}$

Thus $a^2 + \left(\frac{\sqrt{33}}{2a}\right)^2 = 7$

which multiplying by $4a^2$ gives

$(2a^2 - 3)(2a^2 - 11) = 4a^4 + 33 - 28a^2 = 0$.

So, $2a^2 = 3$ or $2a^2 = 11$

i.e. $a = \sqrt{\frac{3}{2}} = \frac{\sqrt{6}}{2}, b = \frac{\sqrt{33}}{\sqrt{6}} = \frac{\sqrt{22}}{2}$

or $a = \sqrt{\frac{11}{2}} = \frac{\sqrt{22}}{2}, b = \frac{\sqrt{6}}{2}$

and so $\sqrt{7+\sqrt{33}} = a + b = \frac{\sqrt{6}+\sqrt{22}}{2}$

Using the same process for the other radicals, we get

$$\sqrt{6+\sqrt{35}} = \frac{\sqrt{10}+\sqrt{14}}{2},$$

so $\sqrt{7+\sqrt{33}} + \sqrt{6+\sqrt{35}}$

$$= \frac{\sqrt{6}+\sqrt{22}+\sqrt{10}+\sqrt{14}}{2} \quad ...(i)$$

and $\sqrt{8+\sqrt{55}} = \dfrac{\sqrt{10}+\sqrt{22}}{2}$,

$$\sqrt{5+\sqrt{21}} = \frac{\sqrt{6}+\sqrt{14}}{2}$$

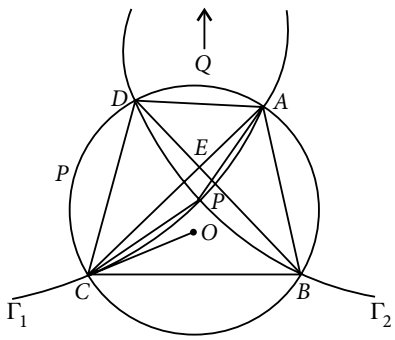
so $\sqrt{8+\sqrt{55}} + \sqrt{5+\sqrt{21}}$

$$= \frac{\sqrt{10}+\sqrt{22}+\sqrt{6}+\sqrt{14}}{2} \quad ...(ii)$$

From (i) (ii)

$$\sqrt{7+\sqrt{33}} + \sqrt{6+\sqrt{35}} = \sqrt{8+\sqrt{55}} + \sqrt{5+\sqrt{21}}.$$

4.



Let the circumcircle of $ABCD$ be Γ. We denote $\angle DAB = \alpha$ and $\angle ABC = \beta$. Then

$$\angle APC = \angle PAB + \angle PCB + \angle ABC = \frac{\pi}{2} + \beta.$$

The locus of the point Γ_1 with the property that $\angle A\Gamma_1 C = \dfrac{\pi}{2} + \beta$ is (an arc of) a circle Γ_1 passing through A and C and touching OC at C.

Since $\angle APC + \angle ACO = \dfrac{\pi}{2} + \beta + \dfrac{\pi}{2} - \beta = \pi$,

OC must be tangent to Γ_1 at C. In the same way, $\angle BPD = \dfrac{\pi}{2} + \alpha$, and the locus of the point Γ_2 with the property that $\angle B\Gamma_2 D = \dfrac{\pi}{2} + \alpha$ is (an arc of) a circle Γ_2 passing through B and D and touching OB at B. Γ_1 and Γ_2 intersect in P and Q (see the figure) but only P has the property that

$$\angle PAB + \angle PCB = \angle PBC + \angle PDC = \frac{\pi}{2} \quad ...(i)$$

Now we'll show that O, E and P are collinear. Consider the circles Γ, Γ_1 and Γ_2. The radical axis of Γ and Γ_1 is AC. The radical axis of Γ and Γ_2 is BD. So E is the radical point of Γ, Γ_1 and Γ_2 [and thus E lies on the radical axis of Γ_1 and Γ_2). O is a point of this axis as well; OC is tangent to Γ_1 and C, and OB is tangent to Γ_2 at B, and $OC = OB$. In other words, OE is the radical axis of Γ_1 and Γ_2, and the intersection points P and Q of these circles are lying on OE. Only P satisfies (i).

5. (i) (a) Let $S_3 = n^2 + (n+1)^2 + (n+2)^2$
$= 3n^2 + 6n + 5 = 3(n^2 + 2n + 1) + 2$.
Thus S_3 is of the form $3k + 2$. But a square of an integer is either of the form $3k$ or $3k + 1$. Therefore S_3 cannot be a square.

(b) Let $S_4 = n^2 + (n+1)^2 + (n+2)^2 + (n+3)^2 = 4n^2 + 12n + 14 = 2(2n^2 + 6n + 7)$.
Because $2n^2 + 6n + 7$ is odd, 2 divides S_4, but 2^2 does not. Therefore S_4 cannot be a square.

(c) Let $S_5 = n^2 + (n+1)^2 + (n+2)^2 + (n+3)^2 + (n+4)^2 = 5n^2 + 20n + 30 = 5(n^2 + 4n + 6)$.
Now $n^2 + 4n + 6 = (n+2)^2 + 2$ is not a multiple of 5 because $(n+2)^2$ is of the form $5k$ or $5k + 1$ or $5k - 1$.
So $(n+2)^2 + 2$ is of the form $5k + 2$, $5k + 3$ or $5k + 1$ and 5 divides S_5 but 5^2 does not. So, S_5 is not a perfect square.

(d) Let $S_6 = n^2 + (n+1)^2 + ... + (n+5)^2$
$= 6n^2 + 30n + 55 = 6(n^2 + 5n + 8) + 7$.
Thus S_6 is of the form $12k + 7$. Since $n^2 + 5n + 8$ is even. But the square of any number is one of the form $12k$, $12k + 1$, $12 + 4$ or $12k + 9$. So S_6 cannot be a square.

(ii) Let $S_{11} = n^2 + (n+1)^2 + ... + (n+10)^2$
$= 11n^2 + 110n + 385 = 11(n^2 + 10n + 35)$.
In order for S_{11} to be a square we must have that $(n+5)^2 + 10 = 11t^2$ so it must be the case that

$$(n+5)^2 - 11t^2 = -10. \qquad (*)$$

The least positive integer solution (n, t) of (*) is $(n, t) = (18, 7)$. The next smallest positive integer solution of (*) is $(n, t) = (38, 13)$. Direct computation verifies that

(a) $18^2 + 19^2 + ... + 28^2 = 5929 = 77^2$ and

(b) $38^2 + 39^2 + ... + 48^2 = 20449 = 143^2$.

■■

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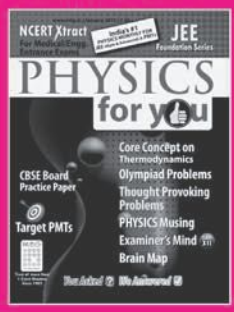

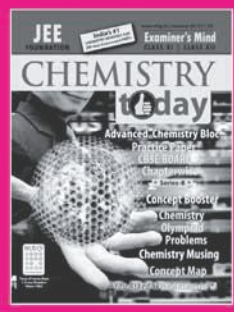

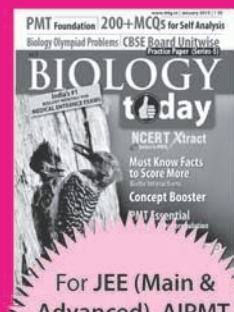

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